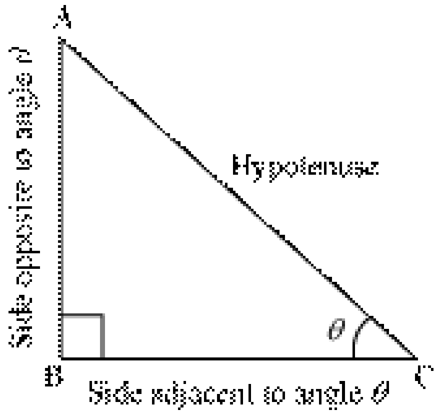
Trigonometry

- **Trigonometric Ratio**



$$\sin\theta = \frac{\text{Opposite side}}{\text{Hypotenuse}} = \frac{AB}{AC}$$

$$\cos\theta = \frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{BC}{AC}$$

$$\tan\theta = \frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{AB}{BC}$$

$$\text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{\text{Hypotenuse}}{\text{Opposite side}} = \frac{AC}{AB}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{Hypotenuse}}{\text{Adjacent side}} = \frac{AC}{BC}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\text{Adjacent side}}{\text{Opposite side}} = \frac{BC}{AB}$$

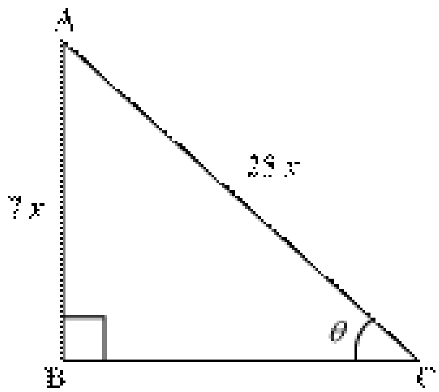
Also, $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$

If one of the trigonometric ratios of an acute angle is known, then the remaining trigonometric ratios of the angle can be calculated.

Example:

If $\sin\theta = \frac{7}{25}$, then find the value of sec θ(1 + tan θ).

Solution:



It is given that $\sin\theta = \frac{7}{25}$

$$\sin\theta = \frac{AB}{AC} = \frac{7}{25}$$

⇒ AB = 7x and AC = 25x, where x is some positive integer
By applying Pythagoras theorem in ΔABC, we get:

$$AB^2 + BC^2 = AC^2$$
$$\Rightarrow (7x)^2 + BC^2 = (25x)^2$$
$$\Rightarrow 49x^2 + BC^2 = 625x^2$$
$$\Rightarrow BC^2 = 625x^2 - 49x^2$$
$$\Rightarrow BC = \sqrt{576}\,x = 24x$$

$$\therefore \sec\theta = \frac{\text{Hypotenuse}}{\text{Adjacent side}} = \frac{25}{24}$$
$$\tan\theta = \frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{7}{24}$$
$$\therefore \sec\theta(1+\tan\theta) = \frac{25}{24}\left(1+\frac{7}{24}\right) = \frac{25}{24}\times\frac{31}{24} = \frac{775}{576}$$

- **Use trigonometric ratio in solving problem.**

Example:

If $\tan\theta = \frac{3}{5}$, then find the value of $\frac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta}$

Solution:

$$\frac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta}$$

Take $\cos\theta$ common from numerator and denominator both

$$= \frac{\frac{\sin\theta}{\cos\theta}+1}{\frac{\sin\theta}{\cos\theta}-1}$$

$$= \frac{\tan\theta+1}{\tan\theta-1}$$

$$= \frac{\frac{3}{5}+1}{\frac{3}{5}-1}$$

$$= \frac{\frac{3+5}{5}}{\frac{3-5}{5}}$$

$$= \frac{8}{-2}$$

$$= -4$$

- **Trigonometric Ratios of some specific angles**

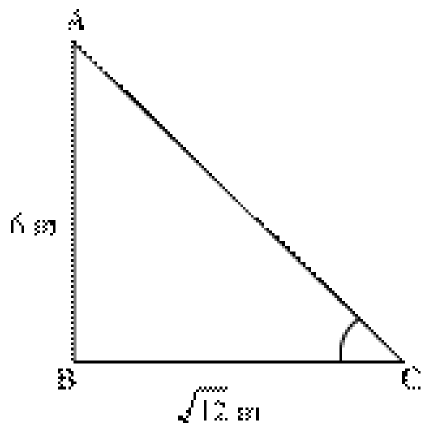
q	**0**	**30°**	**45°**	**60°**	**90°**
sin*q*	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos*q*	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan*q*	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined

cosec*q*	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec*q*	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
cot*q*	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Example 1:

ΔABC is right-angled at B and AB = 6 m, $\mathbf{BC = \sqrt{12}}$ m. Find the measure of ∠A and ∠C.

Solution:



$$\mathbf{AB = 6\ m,}$$

$$\mathbf{BC = \sqrt{12}\ m\ = 2\sqrt{3}\ m}$$

$$\tan C = \frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{AB}{BC} = \frac{6}{2\sqrt{3}} = \sqrt{3}$$

$$\Rightarrow \tan C = \tan 60^\circ \qquad \left[\because \tan 60^\circ = \sqrt{3}\right]$$

$$\Rightarrow \angle C = 60^\circ$$

$$\therefore \angle A = 180^\circ - (90 + 60) = 30^\circ$$

Example 2:

Evaluate the expression

$$4\left(\cos^3 60^\circ - \sin^3 30^\circ\right) + 3\left(\sin 30^\circ - \cos 60^\circ\right)$$

Solution:

$$4\left(\cos^3 60° - \sin^3 30°\right) + 3\left(\sin 30° - \cos 60°\right)$$

$$= 4\left[\left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^3\right] + 3\left(\frac{1}{2} - \frac{1}{2}\right)$$

$$= 4\times 0 + 3\times 0 = 0 + 0 = 0$$

- **Trigonometric Identities**
  1. $\cos^2 A + \sin^2 A = 1$
  2. $1 + \tan^2 A = \sec^2 A$
  3. $1 + \cot^2 A = \text{cosec}^2 A$

Example:

If $\cos\theta = \frac{5}{7}$, find the value of cot θ + cosec θ

Solution:

We have, $\cos\theta = \frac{5}{7}$

$$\text{Now,} \quad \sin^2\theta + \cos^2\theta = 1$$

$$\therefore \sin\theta = \sqrt{1-\cos^2\theta}$$

$$= \sqrt{1-\left(\frac{5}{7}\right)^2}$$

$$= \sqrt{\frac{49-25}{49}} = \frac{2\sqrt{6}}{7}$$

$$\therefore \operatorname{cosec}\theta = \frac{7}{2\sqrt{6}}$$

$$\text{Also,} \ \cot\theta = \frac{\cos\ \theta}{\sin\ \theta}$$

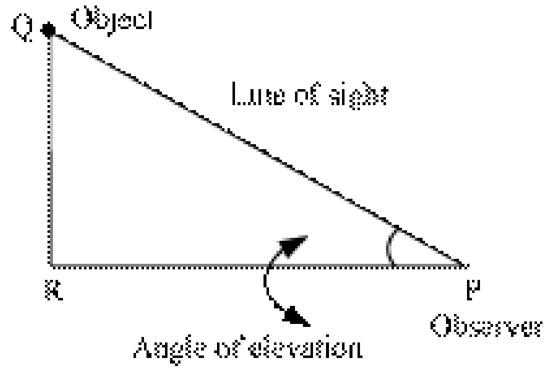
$$= \frac{\frac{5}{7}}{\frac{2\sqrt{6}}{7}} = \frac{5}{2\sqrt{6}}$$

$$\therefore \cot\theta + \operatorname{cosec}\theta = \frac{5}{2\sqrt{6}} + \frac{7}{2\sqrt{6}}$$

$$= \frac{12}{2\sqrt{6}} = \frac{6}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$$

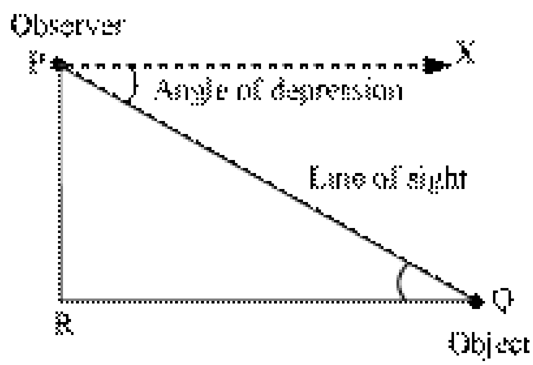
$$= \sqrt{6}$$

- **Some Applications of Trigonometry**
    - **Line of sight:** It is the line drawn from the eye of an observer to a point on the object viewed by the observer.
    - **Angle of Elevation:**



Let P be the position of the eye of the observer. Let Q be the object above the horizontal line PR.
Angle of elevation of the object Q with respect to the observer P is the angle made by the line of sight PQ with the horizontal line PR. That is, ∠QPR is the angle of elevation.

- **Angle of Depression**



Let P be the position of the eye of the observer. Let Q be the object below the horizontal line PX.
Angle of depression of the object Q with respect to the observer P is the angle made by the line of sight PQ with the horizontal line PX. That is, ∠XPQ is the angle of depression. It can be seen that

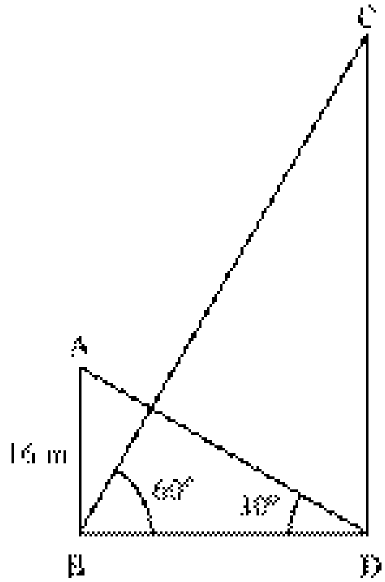
∠PQR = ∠XPQ [Alternate interior angles]

The height or length of an object or the distance between two distant objects can be calculated by using trigonometric ratios.

Example:

The angle of elevation of the top of a tower from the foot of a building is 60° and the angle of elevation of the top of the building from the foot of the tower is 30°. If the building is 16 m tall, then what is the height of the tower?

Solution:



Let AB and CD be the building and the tower respectively.
It is given that, angles of elevation ∠ADB = 30°, ∠CBD = 60°
In ΔABD,

$$\frac{AB}{BD} = \tan 30°$$

$$\Rightarrow \frac{16}{BD} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow BD = 16\sqrt{3}\ m \qquad \text{—(1)}$$

Now, in ΔCBD

$$\frac{CD}{BD} = \tan 60°$$

$$\Rightarrow \frac{CD}{16\sqrt{3}} = \sqrt{3} \qquad [\text{using (1)}]$$

$$\Rightarrow CD = 16\sqrt{3} \times \sqrt{3}\ m = 48\ m$$

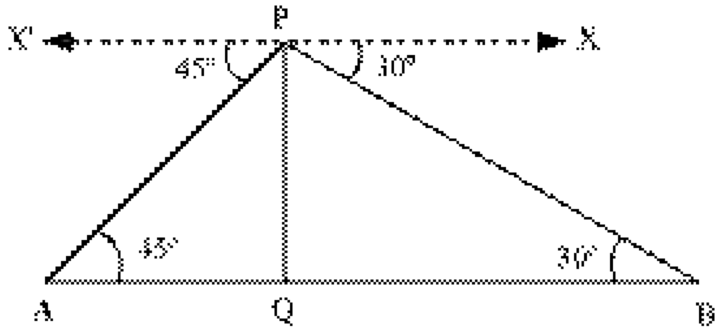
Thus, the height of the tower is 48 m.

Example:

Two wells are located on the opposite sides of a 18 m tall building. As observed from the top of the building, the angles of depression of the two wells are 30° and 45°. Find the distance between the wells. [Use $\sqrt{3} = 1.732$]

Solution:

The given situation can be represented as



Here, PQ is the building. A and B are the positions of the two wells such that:

∠XPB = 30°, ∠XPA =45°

Now, ∠PAQ = ∠XPA = 45°

∠PBQ = ∠XPB = 30°

In ΔPAQ, we have

$$\frac{PQ}{AQ} = \tan 45°$$

$$\Rightarrow \frac{18}{AQ} = 1$$

$$\Rightarrow AQ = 18m$$

In ΔPBQ, we have

$$\frac{PQ}{QB} = \tan 30^\circ$$

$$\Rightarrow \frac{18}{QB} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow QB = 18\sqrt{3}$$

$$\therefore AB = AQ + QB = \left(18 + 18\sqrt{3}\right)m$$

$$= 18\left(1 + \sqrt{3}\right)m$$

$$= 18(1 + 1.732)m$$

$$= 18 \times 2.732 \text{ m}$$

$$= 49.176 \text{ m}$$

- **Trigonometric Ratios of Complementary Angles**

$$\sin\left(90^\circ - \theta\right) = \cos\theta \qquad \cos\left(90^\circ - \theta\right) = \sin\theta$$

$$\tan\left(90^\circ - \theta\right) = \cot\theta \qquad \cot\left(90^\circ - \theta\right) = \tan\theta$$

$$\operatorname{cosec}\left(90^\circ - \theta\right) = \sec\theta \qquad \sec\left(90^\circ - \theta\right) = \operatorname{cosec}\theta$$

Where θ is an acute angle.

Example 1: Evaluate the expression

$$\sin 28^\circ \sin 30^\circ \sin 54^\circ \sec 36^\circ \sec 62^\circ$$

Solution:

$$\sin 28^\circ \sin 30^\circ \sin 54^\circ \sec 36^\circ \sec 62^\circ$$

$$= \left(\sin 28^\circ \sec 62^\circ\right)\left(\sin 54^\circ \sec 36^\circ\right)\sin 30^\circ$$

$$= \left\{\sin 28^\circ \operatorname{cosec}\left(90^\circ - 62^\circ\right)\right\}\left\{\sin 54^\circ \operatorname{cosec}\left(90^\circ - 36^\circ\right)\right\}\sin 30^\circ$$

$$= \left(\sin 28^\circ \operatorname{cosec} 28^\circ\right)\left(\sin 54^\circ \operatorname{cosec} 54^\circ\right)\sin 30^\circ$$

$$= \left(\sin 28^\circ \frac{1}{\sin 28^\circ}\right)\left(\sin 54^\circ \frac{1}{\sin 54^\circ}\right) \times \frac{1}{2}$$

$$= \frac{1}{2}$$

Example 2: Evaluate the expression

$$4\sqrt{3}\left(\sin 40^\circ \sec 30^\circ \sec 50^\circ\right)+\frac{\sin^2 34^\circ+\sin^2 56^\circ}{\sec^2 31^\circ-\cot^2 59^\circ}$$

Solution:

$$4\sqrt{3}\left(\sin 40^\circ \sec 30^\circ \sec 50^\circ\right)+\frac{\sin^2 34^\circ+\sin^2 56^\circ}{\sec^2 31^\circ-\cot^2 59^\circ}$$

$$=4\sqrt{3}\left[\sec 30^\circ\left(\sin 40^\circ \sec 50^\circ\right)\right]+\frac{\sin^2 34^\circ+\sin^2\left(90-56^\circ\right)}{\sec^2 31^\circ-\tan^2\left(90-59^\circ\right)}$$

$$\left[\because \cos\left(90^\circ-\theta\right)=\sin\theta, \tan\left(90^\circ-\theta\right)=\cot\theta\right]$$

$$=4\sqrt{3}\left[\sec 30^\circ \sin 40^\circ \operatorname{cosec}\left(90-50^\circ\right)\right]+\frac{\sin 34^\circ+\cos^2 34^\circ}{\sec^2 31^\circ-\tan^2 31^\circ}$$

$$=4\sqrt{3}\left[\frac{2}{\sqrt{3}}\sin 40^\circ \operatorname{cosec} 40^\circ\right]+\frac{1}{1}$$

$$=8+1=9$$