

Ex 13.1

Q1(i)

We know that $i = \sqrt{-1}$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

In order to find i^n where $n > 4$, we divide n by 4 to get quotient p and remainder q , so that
$$n = 4p + q, 0 \leq q < 4$$

$$\begin{aligned} \text{Then } i^n &= i^{4p+q} \\ &= i^{4p} \times i^q \\ &= (i^4)^p \times i^q \\ &= 1^p \times i^q \\ &= i^q \quad [\because 1^{p-1}] \end{aligned}$$

Hence $i^n = i^q$, where $0 \leq q < 4$

$$\begin{aligned} \therefore i^{457} &= i^{4 \times 114} \times i^1 \\ &= i^1 \\ &= i \end{aligned}$$

Q1(ii)

We know that $i = \sqrt{-1}$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

In order to find i^n where $n > 4$, we divide n by 4 to get quotient p and remainder q , so that
$$n = 4p + q, 0 \leq q < 4$$

$$\begin{aligned} \text{Then } i^n &= i^{4p+q} \\ &= i^{4p} \times i^q \\ &= (i^4)^p \times i^q \\ &= 1^p \times i^q \\ &= i^q \quad [\because 1^{p-1}] \end{aligned}$$

Hence $i^n = i^q$, where $0 \leq q < 4$

$$\begin{aligned} \therefore i^{528} &= i^{4 \times 132} \\ &= (i^4)^{132} \\ &= 1^{132} \\ &= 1 \end{aligned}$$

$$\therefore (i^{528}) = 1$$

Q1(iii)

We know that $i = \sqrt{-1}$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

In order to find i^n where $n > 4$, we divide n by 4 to get quotient p and remainder q , so that
$$n = 4p + q, 0 \leq q < 4$$

$$\begin{aligned} \text{Then } i^n &= i^{4p+q} \\ &= i^{4p} \times i^q \\ &= (i^4)^p \times i^q \\ &= 1^p \times i^q \\ &= i^q \quad [\because 1^{p-1}] \end{aligned}$$

Hence $i^n = i^q$, where $0 \leq q < 4$

$$\begin{aligned} \therefore \frac{1}{i^{58}} &= \frac{1}{i^{4 \times 14} \times i^2} \\ &= \frac{1}{1 \times i^2} \\ &= \frac{1}{-1} \quad [\because i^2 = -1] \\ &= -1 \end{aligned}$$

Q1(iv)

We know that $i = \sqrt{-1}$

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$$i^3 = -i$$

$$i^4 = 1$$

In order to find i^n where $n > 4$, we divide n by 4 to get quotient p and remainder q , so that
$$n = 4p + q, 0 \leq q < 4$$

$$\begin{aligned} \text{Then } i^n &= i^{4p+q} \\ &= i^{4p} \times i^q \\ &= (i^4)^p \times i^q \\ &= 1^p \times i^q \\ &= i^q \quad [\because 1^{p-1}] \end{aligned}$$

Hence $i^n = i^q$, where $0 \leq q < 4$

$$\begin{aligned} \therefore i^{37} + \frac{1}{i^{67}} &= i^{4 \times 9} \times i^1 + \frac{1}{i^{4 \times 16} \times i^3} \\ &= 1 \times i^1 + \frac{1}{1 \times i^3} \\ &= i + \frac{1}{i^3 \times i} \\ &= i + \frac{i}{i^4} \\ &= i + \frac{i}{1} \quad [\because i^4 = 1] \\ &= 2i \end{aligned}$$

$$\therefore i^{37} + \frac{1}{i^{67}} = 2i$$

Q1(v)

We know that $i = \sqrt{-1}$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

In order to find i^n where $n > 4$, we divide n by 4 to get quotient p and remainder q , so that

$$n = 4p + q, 0 \leq q < 4$$

Then $i^n = i^{4p+q}$

$$= i^{4p} \times i^q$$

$$= (i^4)^p \times i^q$$

$$= 1^p \times i^q$$

$$= i^q \quad [\because 1^{p-1}]$$

Hence $i^n = i^q$, where $0 \leq q < 4$

$$\begin{aligned} \left(i^{41} + \frac{1}{i^{257}} \right)^9 &= \left(i^{4 \times 10} \times i^1 + \frac{1}{i^{4 \times 64} \times i^1} \right)^9 \\ &= \left(1 \times i + \frac{1}{1 \times i} \right)^9 \\ &= \left(i + \frac{1}{i} \right)^9 \\ &= \left(i + \frac{1}{i \times i} \times i \right)^9 \\ &= \left(i + \frac{i}{-1} \right)^9 \\ &= (i - i)^9 \\ &= 0 \end{aligned}$$

Q1(vi)

We know that $i = \sqrt{-1}$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

In order to find i^n where $n > 4$, we divide n by 4 to get quotient p and remainder q , so that
$$n = 4p + q, 0 \leq q < 4$$

$$\begin{aligned} \text{Then } i^n &= i^{4p+q} \\ &= i^{4p} \times i^q \\ &= (i^4)^p \times i^q \\ &= 1^p \times i^q \\ &= i^q \quad [\because 1^{p-1}] \end{aligned}$$

Hence $i^n = i^q$, where $0 \leq q < 4$

$$\begin{aligned} (i^{77} + i^{70} + i^{87} + i^{414})^3 &= (i^{4 \times 19} \times i^1 + i^{4 \times 17} \times i^2 + i^{4 \times 21} \times i^3 + i^{4 \times 103} \times i^2)^3 \\ &= (1 \times i + 1 \times i^2 + 1 \times i^3 + 1 \times i^2)^3 \\ &= (i - 1 - i - 1)^3 \\ &= (-2)^3 \\ &= -8 \end{aligned}$$

$$\therefore (i^{77} + i^{70} + i^{87} + i^{414})^3 = -8$$

Q1(vii)

We know that $i = \sqrt{-1}$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

In order to find i^n where $n > 4$, we divide n by 4 to get quotient p and remainder q , so that
$$n = 4p + q, 0 \leq q < 4$$

$$\begin{aligned} \text{Then } i^n &= i^{4p+q} \\ &= i^{4p} \times i^q \\ &= (i^4)^p \times i^q \\ &= 1^p \times i^q \\ &= i^q \quad [\because 1^{p-1}] \end{aligned}$$

Hence $i^n = i^q$, where $0 \leq q < 4$

$$\begin{aligned} \therefore i^{30} + i^{40} + i^{60} &= i^{4 \times 7} \times i^2 + i^{4 \times 10} + i^{4 \times 15} \\ &= 1 \times i^2 + 1 + 1 \\ &= -1 + 1 + 1 \\ &= 1 \end{aligned}$$

$$\therefore i^{30} + i^{40} + i^{60} = 1$$

Q1(viii)

We know that $i = \sqrt{-1}$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

In order to find i^n where $n > 4$, we divide n by 4 to get quotient p and remainder q , so that

$$n = 4p + q, 0 \leq q < 4$$

Then $i^n = i^{4p+q}$

$$= i^{4p} \times i^q$$

$$= (i^4)^p \times i^q$$

$$= 1^p \times i^q$$

$$= i^q \quad [\because 1^{p-1}]$$

Hence $i^n = i^q$, where $0 \leq q < 4$

$$\begin{aligned} i^{49} + i^{68} + i^{89} + i^{110} &= i^{4 \times 12} \times i^1 + i^{4 \times 17} + i^{4 \times 22} \times i^1 + i^{4 \times 27} \times i^2 \\ &= 1 \times i + 1 + 1 \times i + 1 \times i^2 \\ &= i + 1 + i - 1 \\ &= 2i \end{aligned}$$

$$\therefore i^{49} + i^{68} + i^{89} + i^{110} = 2i$$

Q2

$$\begin{aligned} 1 + i^{10} + i^{20} + i^{30} &= 1 + i^{4 \times 2} \times i^2 + i^{4 \times 5} + i^{4 \times 7} \times i^2 \\ &= 1 + 1 \times i^2 + 1 + 1 \times i^2 \\ &= 1 - 1 + 1 - 1 \\ &= 0, \text{ which is real number} \end{aligned}$$

Q3(i)

$$\begin{aligned} i^{49} + i^{68} + i^{89} + i^{110} &= i^{4 \times 12} \times i^1 + i^{4 \times 17} + i^{4 \times 22} \times i^1 + i^{4 \times 27} \times i^2 \\ &= 1 \times i + 1 + 1 \times i + 1 \times i^2 \\ &= i + 1 + i - 1 \\ &= 2i \end{aligned}$$

$$\therefore i^{49} + i^{68} + i^{89} + i^{110} = 2i$$

Q3(ii)

$$\begin{aligned}i^{30} + i^{80} + i^{120} &= i^{4 \times 7} \times i^2 + i^{4 \times 20} + i^{4 \times 30} \\ &= 1 \times i^2 + 1 + 1 \\ &= -1 + 1 + 1 \\ &= 1\end{aligned}$$

$$\therefore i^{30} + i^{80} + i^{120} = 1$$

Q3(iii)

$$\begin{aligned}i + i^2 + i^3 + i^4 &= 1 + (-1) + (-i) + 1 \\ &= 0\end{aligned}$$

$$\therefore i + i^2 + i^3 + i^4 = 0$$

Q3(iv)

$$\begin{aligned}i^5 + i^{10} + i^{15} &= i^{4 \times 1} \times i^1 + i^{4 \times 2} \times i^2 + i^{4 \times 3} \times i^3 \\ &= 1 \times i + 1 \times i^2 + 1 \times i^3 \\ &= i - 1 - i \\ &= -1\end{aligned}$$

$$\therefore i^5 + i^{10} + i^{15} = -1$$

Q3(v)

$$\begin{aligned}\frac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}} &= \frac{i^{4 \times 148} + i^{147} \times i^2 + i^{4 \times 147} + i^{4 \times 146} \times i^2 + i^{4 \times 146}}{i^{4 \times 145} \times i^2 + i^{4 \times 145} + i^{4 \times 144} \times i^2 + i^{4 \times 144} + i^{4 \times 143} \times i^2} \\ &= \frac{1 + 1 \times i^2 + 1 + 1 \times i^2 + 1}{1 \times i^2 + 1 + 1 \times i^2 + 1 + 1 \times i^2} \\ &= \frac{1 - 1 + 1 - 1 + 1}{-1 + 1 - 1 + 1 - 1} \\ &= \frac{1}{-1} \\ &= -1\end{aligned}$$

$$\therefore \frac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}} = -1$$

Q3(vi)

$$\begin{aligned}
& 1 + i^2 + i^4 + i^6 + i^8 + \dots + i^{20} \\
&= 1 + i^2 + i^4 + i^{4 \times 1} \times i^2 + i^{4 \times 2} + i^{4 \times 2} \times i^2 + i^{4 \times 3} + i^{4 \times 3} \times i^2 + i^{4 \times 4} + i^{4 \times 4} \times i^2 + i^{4 \times 5} \\
&= 1 - 1 + 1 + 1 \times i^2 + 1 \\
&= 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 \\
&= 1
\end{aligned}$$

Q3(vii)

$$\begin{aligned}
(1+i)^6 + (1-i)^3 &= [(1+i)^2]^3 + (1-i)^3 \\
&= (1+i^2+2i)^3 + (1-3+3i^2-i^3) \\
&= (1-1+2i)^3 + (1-3-3+i) \\
&= 8i^3 - 2 - 2i \\
&= -8 - 2 - 2i \\
&= -2 - 10i
\end{aligned}$$

Ex 13.2

Q1(i)

$$\begin{aligned}(1+i)(1+2i) &= 1 \times (1+2i) + i(1+2i) \\ &= 1+2i+i+2i^2 \\ &= 1+3i-2 \\ &= -1+3i\end{aligned}$$

$$\therefore (1+i)(1+2i) = -1+3i$$

Q1(ii)

$$\begin{aligned}\frac{3+2i}{-2+i} &= \frac{3+2i}{-2+i} \times \frac{-2-i}{-2-i} && \text{[Rationalising the denominator]} \\ &= \frac{3(-2-i)+2i(-2-i)}{(-2)^2 - (i)^2} && [\because (a+ib)(a-ib) = a^2 + b^2] \\ &= \frac{-6-3i-4i+2}{4+1} && [\because -i^2 = 1] \\ &= \frac{-4-7i}{5} \\ &= \frac{-4}{5} - \frac{7}{5}i\end{aligned}$$

$$\therefore \frac{3+2i}{-2+i} = \frac{-4}{5} - \frac{7}{5}i$$

Q1(iii)

$$\begin{aligned}\frac{1}{(2+i)^2} &= \frac{1}{2^2 + (i)^2 + 2 \times 2 \times i} \\ &= \frac{1}{4-1+4i} \\ &= \frac{1}{3+4i} \\ &= \frac{1}{(3+4i)} \times \frac{(3-4i)}{(3-4i)} && \text{[on rationalising the denominator]} \\ &= \frac{3-4i}{3^2+4^2} && [\because (a+ib)(a-ib) = a^2 + b^2] \\ &= \frac{3-4i}{25} \\ &= \frac{3}{25} - \frac{4}{25}i\end{aligned}$$

$$\therefore \frac{1}{(2+i)^2} = \frac{3}{25} - \frac{4}{25}i$$

Q1(iv)

$$\frac{1-i}{1+i} = \frac{(1-i)}{(1+i)} \times \frac{(1-i)}{(1-i)} \quad (\text{Rationalising the denominator})$$

$$= \frac{(1-i)^2}{1^2+1^2} \quad [\because (a+ib)(a-ib) = a^2 + b^2]$$

$$= \frac{1^2+i^2-2 \times i \times 1}{2}$$

$$= \frac{-2i}{2}$$

$$= -i$$

$$= 0 - i$$

$$\therefore \frac{1-i}{1+i} = 0 - i$$

Q1(v)

$$\frac{(2+i)^3}{2+3i} = \frac{2^3+i^3+3 \times 2 \times i(2+i)}{2+3i} \quad [\because (a+b)^3 = a^3 + b^3 + 3ab(a+b)]$$

$$= \frac{(8-i+6i(2+i))}{2+3i} \times \frac{(2-3i)}{2-3i} \quad (\text{On rationalising the denominator})$$

$$= \frac{(8-i+12i+6i^2)(2-3i)}{2^2+3^2}$$

$$= \frac{(8-6+11i)(2-3i)}{4+9} \quad (\because i^2 = -1)$$

$$= \frac{(2+11i)(2-3i)}{13}$$

$$= \frac{4-6i+22i+33}{13}$$

$$= \frac{37+16i}{13}$$

$$= \frac{37}{13} + \frac{16}{13}i$$

$$\therefore \frac{(2+i)^3}{2+3i} = \frac{37}{13} + \frac{16}{13}i$$

Q1(vi)

$$\begin{aligned}
\frac{(1+i)(1+\sqrt{3}i)}{1-i} &= \frac{1(1+\sqrt{3}i) + i(1+\sqrt{3}i)}{1-i} \\
&= \frac{(1+\sqrt{3}i+i-\sqrt{3})}{1-i} \quad (\because i^2 = -1) \\
&= \frac{(1-\sqrt{3})+i(1+\sqrt{3})}{1-i} \times \frac{(1+i)}{1+i} \quad (\text{Rationalising the denominator}) \\
&= \frac{(1-\sqrt{3})(1+i) + i(1+\sqrt{3})(1+i)}{1^2+1^2} \\
&= \frac{1+i-\sqrt{3}(1+i) + i(1+i+\sqrt{3}(1+i))}{2} \\
&= \frac{1+i-\sqrt{3}-\sqrt{3}i+i(1+i+\sqrt{3}+\sqrt{3}i)}{2} \\
&= \frac{1-\sqrt{3}+i-\sqrt{3}i+i-1+\sqrt{3}i-\sqrt{3}}{2} \\
&= \frac{-2\sqrt{3}+2i}{2} \\
&= -\sqrt{3}+i
\end{aligned}$$

$$\therefore \frac{(1+i)(1+\sqrt{3}i)}{1-i} = -\sqrt{3}+i$$

Q1(vii)

$$\begin{aligned}
\frac{2+3i}{4+5i} &= \frac{2+3i}{4+5i} \times \frac{(4-5i)}{(4-5i)} \quad (\text{rationalising the denominator}) \\
&= \frac{2(4-5i)+3i(4-5i)}{4^2+5^2} \\
&= \frac{8-10i+12i+15}{16+25} \quad (\because i^2 = -1) \\
&= \frac{23+2i}{41} \\
&= \frac{23}{41} + \frac{2}{41}i
\end{aligned}$$

$$\therefore \frac{2+3i}{4+5i} = \frac{23}{41} + \frac{2}{41}i$$

Q1(viii)

$$\begin{aligned} \frac{(1-i)^3}{1-i^3} &= \frac{1^3 - i^3 - 3 \times 1 \times i(1-i)}{1 - (-i)} && \left[\begin{array}{l} \because (a-b)^3 = a^3 - b^3 - 3ab(a-b) \\ \text{and } i^3 = -i \end{array} \right] \\ &= \frac{1 - (-i) - 3i(1-i)}{1+i} \\ &= \frac{1+i-3i-3}{1+i} \\ &= \frac{-2-2i}{1+i} \\ &= \frac{-2(1+i)}{1+i} \\ &= -2 \\ &= -2+0i \end{aligned}$$

$$\therefore \frac{(1-i)^3}{1-i^3} = -2+0i$$

Q1(ix)

$$\begin{aligned} (1+2i)^{-3} &= \frac{1}{(1+2i)^3} && \left(\because z^{-3} = \frac{1}{z^3} \right) \\ &= \frac{1}{1^3 + (2i)^3 + 3 \times 1 \times 2i(1+2i)} \\ &= \frac{1}{1^3 + 2^3 \times i^3 + 6i(1+2i)} \\ &= \frac{1}{1-8i+6i-12} && \left(\because i^3 = -i \text{ and } i^2 = -1 \right) \\ &= \frac{1}{-11-2i} \\ &= \frac{1}{-11-2i} \times \frac{(-11+2i)}{(-11+2i)} \\ &= \frac{-11+2i}{(-11)^2 + 2^2} \\ &= \frac{-11+2i}{121+4} \\ &= \frac{-11}{125} + \frac{2}{125}i \end{aligned}$$

$$\therefore (1+2i)^{-3} = \frac{-11}{125} + \frac{2}{125}i$$

Q1(x)

$$\begin{aligned}\frac{3-4i}{(4-2i)(1+i)} &= \frac{3-4i}{4(1+i)-2i(1+i)} \\ &= \frac{3-4i}{4+4i-2i+2} \\ &= \frac{3-4i}{6+2i} \\ &= \frac{3-4i}{6+2i} \times \frac{6-2i}{6-2i} \\ &= \frac{3(6-2i)-4i(6-2i)}{6^2+2^2} \\ &= \frac{18-6i-24i-8}{36+4} \\ &= \frac{10-30i}{40} \\ &= \frac{10(1-3i)}{40} \\ &= \frac{1-3i}{4} \\ &= \frac{1}{4} - \frac{3}{4}i\end{aligned}$$

$$\therefore \frac{3-4i}{(4-2i)(1+i)} = \frac{1}{4} - \frac{3}{4}i$$

Q1(xi)

$$\begin{aligned}\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right) &= \frac{(1+i-2(1-4i))}{(1-4i)(1+i)} \times \frac{3-4i}{5+i} \\ &= \frac{(1+i-2+8i)}{1(1+i)-4i(1+i)} \times \frac{3-4i}{5+i} \\ &= \frac{(1+i-2+8i)}{1(1+i)-4i(1+i)} \times \frac{3-4i}{5+i} \\ &= \frac{(-1+9i)}{(1+i-4i+4)} \times \frac{3-4i}{5+i} \\ &= \frac{-1(3-4i)+9i(3-4i)}{(5-3i)(5+i)} \\ &= \frac{-3+4i+27i+36}{5(5+i)-3i(5+i)} \\ &= \frac{33+31i}{25+5i-15i+3} \\ &= \frac{33+31i}{28-10i} \\ &= \frac{(33+31i)}{28-10i} \times \frac{(28+10i)}{28+10i} \\ &= \frac{33 \times 28 + 33 \times 10i + 31i \times 28 + 31i \times 10i}{28^2 + 10^2} \\ &= \frac{924 + 330i + 868i - 310}{784 + 100} \\ &= \frac{614 + 1198i}{884} \\ &= \frac{614}{884} + \frac{1198}{884}i \\ &= \frac{307}{442} + \frac{599}{442}i\end{aligned}$$

$$\therefore \left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right) = \frac{307}{442} + \frac{599}{442}i$$

Q1(xii)

We have

$$\begin{aligned}\frac{5+\sqrt{2}i}{1-\sqrt{2}i} &= \frac{5+\sqrt{2}i}{1-\sqrt{2}i} \times \frac{1+\sqrt{2}i}{1+\sqrt{2}i} \\ &= \frac{5(1+\sqrt{2}i)+\sqrt{2}i(1+\sqrt{2}i)}{1+2} \\ &= \frac{5+5\sqrt{2}i+\sqrt{2}i-2}{3} \\ &= \frac{3+6\sqrt{2}i}{3} \\ &= 1+2\sqrt{2}i\end{aligned}$$

Therefore, $\frac{5+\sqrt{2}i}{1-\sqrt{2}i} = 1+2\sqrt{2}i$

Q2(i)

We have $(x + iy)(2 - 3i) = 4 + i$

$$\Rightarrow x(2 - 3i) + iy(2 - 3i) = 4 + i$$

$$\Rightarrow 2x - 3xi + 2yi + 3y = 4 + i$$

$$\Rightarrow 2x + 3y + i(-3x + 2y) = 4 + i$$

Equating the real and imaginary parts we get

$$2x + 3y = 4 \dots\dots\dots(i)$$

$$-3x + 2y = 1 \dots\dots\dots(ii)$$

Multiplying (i) by 3 and (ii) by 2 and adding

$$6x - 6x - 9y + 4y = 12 + 2$$

$$\Rightarrow 13y = 14$$

$$\Rightarrow y = \frac{14}{13}$$

Substituting the value of y in (i), we get

$$2x + 3 \times \frac{14}{13} = 4$$

$$\Rightarrow 2x + \frac{42}{13} = 4$$

$$\Rightarrow 2x = 4 - \frac{42}{13}$$

$$\Rightarrow 2x = \frac{52 - 42}{13}$$

$$\Rightarrow 2x = \frac{10}{13}$$

$$\Rightarrow x = \frac{5}{13}$$

Hence

$$x = \frac{5}{13} \text{ and } y = \frac{14}{13}$$

Q2(ii)

$$(3x - 2iy)(2 + i)^2 = 10(1 + i)$$

$$\Rightarrow (3x - 2iy)(2^2 + i^2 + 2 \times 2 \times i) = 10 + 10i$$

$$\Rightarrow (3x - 2iy)(4 - 1 + 4i) = 10 + 10i$$

$$\Rightarrow 3x(3 + 4i) - 2iy(3 + 4i) = 10 + 10i$$

$$\Rightarrow 9x + 12xi - 6yi + 8y = 10 + 10i$$

$$\Rightarrow 9x + 8y + i(12x - 6y) = 10 + 10i$$

Equating the real and imaginary parts we get

$$9x + 8y = 10 \dots\dots\dots (i)$$

$$12x - 6y = 10 \dots\dots\dots (ii)$$

Multiplying (i) by 6 and (ii) by 8 and adding

$$54x + 96y + 48y - 48y = 60 + 80$$

$$\Rightarrow 150x = 140$$

$$\Rightarrow x = \frac{140}{150}$$

$$\Rightarrow x = \frac{14}{15}$$

Substituting value of x in (i) we get

$$9 \times \frac{14}{15} + 8y = 10$$

$$\Rightarrow \frac{42}{5} + 8y = 10$$

$$\Rightarrow 8y = 10 - \frac{42}{5}$$

$$\Rightarrow 8y = \frac{50 - 42}{5}$$

$$\Rightarrow 8y = \frac{8}{5}$$

$$\Rightarrow y = \frac{1}{5}$$

Q2(iii)

$$\begin{aligned}\frac{(1+i)x - 2i}{3+i} + \frac{(2-3i)y + i}{3-i} &= i \\ \Rightarrow \frac{(3-i)((1+i)x - 2i) + (3+i)((2-3i)y + i)}{(3+i)(3-i)} &= i \\ \Rightarrow \frac{(3-i)(1+i)x - 2i(3-i) + (3+i)(2-3i)y + i(3+i)}{3^2+1^2} &= i \\ \Rightarrow \frac{(3+3i-i+1)x - 6i - 2 + (6-9i+2i+3)y + 3i - 1}{9+1} &= i \\ \Rightarrow \frac{(4+2i)x - 6i - 2 + (9-7i)y + 3i - 1}{10} &= i \\ \Rightarrow 4x + 2ix - 6i - 2 + 9y - 7iy + 3i - 1 &= 10i \\ \Rightarrow 4x + 9y - 3 + i(2x - 7y - 3) &= 10i\end{aligned}$$

Equating real and imaginary parts we get

$$4x + 9y - 3 = 0 \dots\dots\dots (i)$$

$$\text{and } 2x - 7y - 3 = 10$$

$$i.e. 2x - 7y = 13 \dots\dots\dots (ii)$$

Multiplying (i) by 7, (ii) by 9 and adding we get

$$28x + 18x + 63y - 63y = 117 + 21$$

$$\Rightarrow 46x = 117 + 21$$

$$\Rightarrow 46x = 138$$

$$\begin{aligned}\Rightarrow x &= \frac{138}{46} \\ &= 3\end{aligned}$$

Substituting the value of $x = 3$ in (i), we get

$$4 \times 3 + 9y = 3$$

$$\Rightarrow 9y = -9$$

$$\Rightarrow y = \frac{-9}{9}$$

$$\Rightarrow y = -1$$

Hence

$$x = 3, y = -1$$

Q2(iv)

$$(1+i)(x+iy) = 2-5i$$

$$\Rightarrow 1(x+iy) + i(x+iy) = 2-5i$$

$$\Rightarrow x+iy+ix-y = 2-5i$$

$$\Rightarrow x-y+i(x+y) = 2-5i$$

Equating real and imaginary parts we get

$$x-y = 2 \dots\dots\dots (i)$$

$$x+y = -5 \dots\dots\dots (ii)$$

Adding (i) and (ii) we get

$$2x = 2-5$$

$$\Rightarrow 2x = -3$$

$$\Rightarrow x = \frac{-3}{2}$$

Substituting the value of x in (i), we get

$$\frac{-3}{2} - y = 2$$

$$\Rightarrow \frac{-3}{2} - 2 = y$$

$$\Rightarrow y = \frac{-3-4}{2}$$

$$\Rightarrow y = \frac{-7}{2}$$

Hence

$$x = \frac{-3}{2}, y = \frac{-7}{2}$$

Q3(i)

If $z = x + iy$ is a complex number, then the conjugate of z denoted by \bar{z} is defined as $\bar{z} = x - iy$

$$\text{let } z = 4 - 5i$$

$$\Rightarrow \bar{z} = 4 + 5i$$

Q3(ii)

$$\begin{aligned}\text{let } z &= \frac{1}{3+5i} \\ &= \frac{1}{3+5i} \times \frac{(3-5i)}{(3-5i)} \quad (\text{Onrationalising the denominator}) \\ &= \frac{3-5i}{3^2+5^2} \\ \Rightarrow z &= \frac{3-5i}{9+25}\end{aligned}$$

$$\begin{aligned}\text{So } \bar{z} &= \frac{3+5i}{34} \\ &= \frac{3}{34} + \frac{5}{34}i\end{aligned}$$

Q3(iii)

$$\begin{aligned}\text{let } z &= \frac{1}{1+i} \\ &= \frac{1}{1+i} \times \frac{(1-i)}{(1-i)} \\ &= \frac{1-i}{1^2+1^2} \\ &= \frac{1-i}{2}\end{aligned}$$

$$\begin{aligned}\therefore \bar{z} &= \frac{1+i}{2} \\ &= \frac{1}{2} + \frac{1}{2}i\end{aligned}$$

Q3(iv)

$$\begin{aligned}\text{let } z &= \frac{(3-i)^2}{2+i} \\ &= \frac{3^2 + i^2 - 2 \times 3 \times i}{2+i} \\ &= \frac{9 - 1 - 6i}{2+i} \\ &= \frac{8 - 6i}{2+i} \\ &= \frac{8-6i}{2+i} \times \frac{2-i}{2-i} \\ &= \frac{8(2-i) - 6i(2-i)}{2^2 + 1^2} \\ &= \frac{16 - 8i - 12i - 6}{4+1} \\ &= \frac{10 - 20i}{5} \\ \Rightarrow z &= 2 - 4i\end{aligned}$$

Hence

$$\bar{z} = 2 + 4i$$

Q3(v)

$$\begin{aligned}\text{let } z &= \frac{(1+i)(2+i)}{3+i} \\ &= \frac{2+i+i(2+i)}{3+i} \\ &= \frac{2+i+2i-1}{3+i} \\ &= \frac{1+3i}{3+i} \\ &= \frac{(1+3i)}{(3+i)} \times \frac{(3-i)}{(3-i)} \\ &= \frac{3-i+3i(3-i)}{3^2+1^2} \\ &= \frac{3-i+9i+3}{9+1} \\ &= \frac{6+8i}{10} \\ &= \frac{2(3+4i)}{10} \\ \Rightarrow z &= \frac{3+4i}{5}\end{aligned}$$

Hence

$$\begin{aligned}\bar{z} &= \frac{3-4i}{5} \\ &= \frac{3}{5} - \frac{4}{5}i\end{aligned}$$

Q3(vi)

$$\begin{aligned}\text{let } z &= \frac{(3-2i)(2+3i)}{(1+2i)(2-i)} \\ &= \frac{3(2+3i) - 2i(2+3i)}{2-i+2i(2-i)} \\ &= \frac{6+9i-4i+6}{2-i+4i+2} \\ &= \frac{12+5i}{4+3i} \\ &= \frac{12+5i}{4+3i} \times \frac{4-3i}{4-3i} \\ &= \frac{12(4-3i)+5i(4-3i)}{4(4-3i)+3i(4-3i)} \\ &= \frac{48-36i+20i+15}{16-12i+12i+9} \\ &= \frac{63-16i}{16+9} \\ \Rightarrow z &= \frac{63-16i}{25}\end{aligned}$$

$$\begin{aligned}\therefore \bar{z} &= \frac{63+16i}{25} \\ &= \frac{63}{25} + \frac{16}{25}i\end{aligned}$$

Q4(i)

If $z = x + iy$ is a complex number, then the multiplicative inverse of z , denoted by z^{-1} or $\frac{1}{z}$

$$\begin{aligned}\text{is defined as } z^{-1} &= \frac{1}{z} \\ &= \frac{1}{x+iy} \\ &= \frac{1}{x+iy} \times \frac{x-iy}{x-iy} \\ &= \frac{x-iy}{x^2+y^2} \\ &= \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i\end{aligned}$$

Given

$$\begin{aligned}z &= 1-i \\ \therefore z^{-1} &= \frac{1}{1^2+1^2} - \frac{(-1)}{1^2+1^2} \times i \\ &= \frac{1}{2} + \frac{1}{2}i\end{aligned}$$

Q4(ii)

$$\begin{aligned}\text{let } z &= (1+i\sqrt{3})^2 \\ &= 1^2 + (i\sqrt{3})^2 + 2 \times 1 \times i\sqrt{3} \\ &= 1 - 3 + 2\sqrt{3}i \\ &= -2 + 2\sqrt{3}i\end{aligned}$$

$$\begin{aligned}\therefore z^{-1} &= \frac{-2}{(-2)^2 + (2\sqrt{3})^2} - \frac{2\sqrt{3}i}{(-2)^2 + (2\sqrt{3})^2} \\ &= \frac{-2}{4+12} - \frac{2\sqrt{3}i}{4+12} \\ &= \frac{-2}{16} - \frac{2\sqrt{3}i}{16} \\ &= \frac{-1}{8} - \frac{\sqrt{3}i}{8}\end{aligned}$$

Q4(iii)

$$\begin{aligned}\text{let } z &= 4 - 3i \\ \text{Then } z^{-1} &= \frac{4}{4^2 + (-3)^2} - \frac{(-3)}{4^2 + (-3)^2} \\ &= \frac{4}{16+9} + \frac{3}{16+9}i \\ &= \frac{4}{25} + \frac{3}{25}i\end{aligned}$$

Q4(iv)

$$\begin{aligned}\text{let } z &= \sqrt{5} + 3i \\ \text{Then } z^{-1} &= \frac{\sqrt{5}}{(\sqrt{5})^2 + (3)^2} - \frac{3}{(\sqrt{5})^2 + (3)^2}i \\ &= \frac{\sqrt{5}}{5+9} - \frac{3}{5+9}i \\ &= \frac{\sqrt{5}}{14} - \frac{3}{14}i\end{aligned}$$

Q5

If $z = x + iy$ then $|z| = \sqrt{x^2 + y^2}$

We have,

$$z_1 = 2 - i, z_2 = 1 + i$$

$$\begin{aligned} z_1 + z_2 &= 2 - i + 1 + i \\ &= 3 \end{aligned}$$

$$\begin{aligned} \text{And } z_1 - z_2 &= 2 - i - 1 - i \\ &= 1 - 2i \end{aligned}$$

$$\begin{aligned} \frac{z_1 + z_2 + 1}{z_1 - z_2 + i} &= \frac{3 + 1}{1 - 2i + i} \\ &= \frac{4}{1 - i} \\ &= \frac{4}{1 - i} \times \frac{1 + i}{1 + i} \\ &= \frac{4(1 + i)}{1^2 + 1^2} \\ &= \frac{4(1 + i)}{2} \\ &= 2(1 + i) \end{aligned}$$

$$\begin{aligned} \therefore \left| \frac{z_1 + z_2 + 1}{z_1 - z_2 + i} \right| &= |2(1 + i)| \\ &= 2|1 + i| \\ &= 2 \times \sqrt{1^2 + 1^2} \\ &= 2 \times \sqrt{2} \\ &= 2\sqrt{2} \end{aligned}$$

$$(\because |z_1 z_2| = |z_1| \times |z_2|)$$

Q6

(i)

$$\frac{z_1 z_2}{z_1} = \frac{z_1 z_2}{z_1} \times \frac{z_1}{z_1} \quad (\text{rationalising the denominator})$$

$$= \frac{(z_1)^2 z_2}{z_1 z_1}$$

$$= \frac{(2-i)^2 (-2+i)}{|z_1|^2} \quad (\because z \bar{z} = |z|^2)$$

$$= \frac{(2^2 + i^2 - 2 \times 2 \times i) (-2+i)}{|2-i|^2}$$

$$= \frac{(4 - 1 - 4i) (-2+i)}{2^2 + (-1)^2}$$

$$= \frac{(3 - 4i) (-2+i)}{4+i}$$

$$= 3(-2+i) - 4i(-2+i)$$

$$= \frac{-6 + 3i + 8i + 4}{5}$$

$$= \frac{-2 + 11i}{5}$$

$$\begin{aligned} \therefore \operatorname{Re}\left(\frac{z_1 z_2}{z_1}\right) &= \operatorname{Re}\left(\frac{-2}{5} + \frac{11}{5}i\right) \\ &= \frac{-2}{5} \end{aligned}$$

(ii)

$$\frac{1}{z_1 z_1} = \frac{1}{|z_1|^2}$$

$$= \frac{1}{|2-i|^2}$$

$$= \frac{1}{2^2 + (-1)^2}$$

$$= \frac{1}{4+1}$$

$$= \frac{1}{5}, \text{ which is purely real}$$

$$\therefore \operatorname{Im}\left(\frac{1}{z_1 z_1}\right) = 0$$

Q7

$$\begin{aligned}
\text{let } z &= \frac{1+i}{1-i} - \frac{1-i}{1+i} \\
&= \frac{(1+i)^2 - (1-i)^2}{(1-i)(1+i)} \\
&= \frac{1^2 + i^2 + 2 \times 1 \times i - (1^2 + i^2 - 2 \times 1 \times i)}{1^2 + 1^2} \\
&= \frac{1 - 1 + 2i - (1 - 1 - 2i)}{2} \\
&= \frac{2i + 2i}{2} \\
&= \frac{4i}{2} \\
\Rightarrow z &= 2i
\end{aligned}$$

$$\begin{aligned}
\therefore |z| &= |2i| \\
&= 2|i| && (\because |z_1 z_2| = |z_1| \times |z_2|) \\
&= 2 \times 1 && (\because |i| = 1) \\
&= 2
\end{aligned}$$

Q8

$$x + iy = \frac{a + ib}{a - ib}$$

$$\Rightarrow \overline{(x + iy)} = \overline{\left(\frac{a + ib}{a - ib}\right)} \quad (\text{on taking conjugate both sides})$$

$$\begin{aligned}
\Rightarrow x - iy &= \frac{\overline{(a + ib)}}{\overline{(a - ib)}} && \left(\because \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}} \right) \\
&= \frac{a - ib}{a + ib}
\end{aligned}$$

$$\because (x + iy)(x - iy) = \frac{a + ib}{a - ib} \times \frac{a - ib}{a + ib}$$

$$\Rightarrow x^2 + y^2 = 1$$

proved

Q9

For $n = 1$, we have,

$$\begin{aligned}\left(\frac{1+i}{1-i}\right)^1 &= \frac{1+i}{1-i} \\ &= \frac{1+i}{1-i} \times \frac{1+i}{1+i} \\ &= \frac{(1+i)^2}{1^2+1^2} \\ &= \frac{1^2+i^2+2 \times 1 \times i}{2} \\ &= \frac{2i}{2} \quad (\because i^2 = -1) \\ &= i, \text{ which is not real}\end{aligned}$$

For $n = 2$, we have

$$\begin{aligned}\left(\frac{1+i}{1-i}\right)^2 &= i^2 \quad \left(\because \frac{1+i}{1-i} = 1 \text{ form above}\right) \\ &= -1, \text{ which is real}\end{aligned}$$

Hence the least positive integral value of n is 2.

Q10

$$\begin{aligned}\text{let } z &= \frac{1+i \cos \theta}{1-2i \cos \theta} \\ &= \frac{1+i \cos \theta}{1-2i \cos \theta} \times \frac{1+2i \cos \theta}{1+2i \cos \theta} \\ &= \frac{1+2i \cos \theta + i \cos \theta (1+2i \cos \theta)}{1^2 + (2 \cos \theta)^2} \\ &= \frac{1+2i \cos \theta + i \cos \theta - 2 \cos^2 \theta}{1+4 \cos^2 \theta} \\ &= \frac{1-2 \cos^2 \theta + 3i \cos \theta}{1+4 \cos^2 \theta} \\ &= \frac{1-2 \cos^2 \theta}{1+4 \cos^2 \theta} + \frac{3 \cos \theta}{1+4 \cos^2 \theta} i\end{aligned}$$

we know that z is purely real if and only if $\text{Im} z = 0$

$$\begin{aligned}\therefore \frac{3 \cos \theta}{1+4 \cos^2 \theta} &= 0 \quad (\because z \text{ is given to be purely real}) \\ \Rightarrow 3 \cos \theta &= 0 \\ \Rightarrow \cos \theta &= 0 \\ \Rightarrow \cos \theta &= \cos \frac{\pi}{2}\end{aligned}$$

\therefore The general solution is given by

$$\theta = 2n\pi \pm \frac{\pi}{2}, n \in Z$$

Q11

$$\begin{aligned}\text{let } z &= \frac{(1+i)^n}{(1-i)^{n-2}} \\ &= \frac{(1+i)^n}{(1-i)^n} (1-i)^2 \\ &= \left(\frac{1+i}{1-i}\right)^n \times (1-i)^2 \\ &= i^n (1+i^2 - 2 \times 1 \times i) \quad \left(\because \frac{1+i}{1-i} = i, \text{ using problem 10}\right) \\ &= i^n (1-1-2i) \\ &= -2i \times i^n \\ &= -2i^{n+1}\end{aligned}$$

\therefore For $n = 1$

$$\begin{aligned}z &= -2i^{1+1} \\ &= -2i^2 \\ &= 2, \text{ which is a real number}\end{aligned}$$

\therefore The smallest positive integer value of n is 1.

Q12

$$\begin{aligned}\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 &= x + iy \\ \Rightarrow \left(\frac{(1+i)(1+i)}{(1-i)(1+i)}\right)^3 - \left(\frac{(1-i)(1-i)}{(1+i)(1-i)}\right)^3 &= x + iy \text{ [Rationalizing the denominator]} \\ \Rightarrow \left(\frac{1+2i-1}{1+1}\right)^3 - \left(\frac{1-2i-1}{1+1}\right)^3 &= x + iy \\ \Rightarrow \left(\frac{2i}{2}\right)^3 - \left(\frac{-2i}{2}\right)^3 &= x + iy \\ \Rightarrow i^3 - (-i)^3 &= x + iy \\ \Rightarrow -i - i &= x + iy \\ \Rightarrow -2i &= x + iy\end{aligned}$$

Comparing the real and imaginary parts,

$$(x, y) = (0, 2)$$

Q13

$$\frac{(1+i)^2}{2-i} = x + iy$$

$$\Rightarrow \frac{(1+2i-1)}{2-i} = x + iy$$

$$\Rightarrow \frac{2i}{2-i} = x + iy$$

$$\Rightarrow \frac{2i(2+i)}{(2-i)(2+i)} = x + iy \text{ [Rationalizing the denominator]}$$

$$\Rightarrow \frac{2(2i-1)}{4+1} = x + iy$$

$$\Rightarrow \frac{4i-2}{5} = x + iy$$

$$\Rightarrow -\frac{2}{5} + i\frac{4}{5} = x + iy$$

Comparing the real and imaginary parts, we get

$$x = -\frac{2}{5}, y = \frac{4}{5}$$

$$x + y = \frac{2}{5}$$

Q14

$$\left(\frac{1-i}{1+i}\right)^{100} = a + ib$$

$$\Rightarrow \left(\frac{(1-i)(1-i)}{(1+i)(1-i)}\right)^{100} = a + ib \text{ [Rationalizing the denominator]}$$

$$\Rightarrow \left(\frac{(1-2i-1)}{(1+1)}\right)^{100} = a + ib$$

$$\Rightarrow \left(\frac{-2i}{2}\right)^{100} = a + ib$$

$$\Rightarrow (-i)^{100} = a + ib$$

$$\Rightarrow 1 = a + ib$$

Comparing, we get $(a,b) = (1,0)$

Q15

$$a = \cos \theta + i \sin \theta$$

$$\frac{1+a}{1-a}$$

$$= \frac{1 + \cos \theta + i \sin \theta}{1 - \cos \theta - i \sin \theta}$$

$$= \frac{(1 + \cos \theta + i \sin \theta)(1 - \cos \theta + i \sin \theta)}{(1 - \cos \theta - i \sin \theta)(1 - \cos \theta + i \sin \theta)}$$

$$= \frac{(1 + \cos \theta + i \sin \theta)(1 - \cos \theta + i \sin \theta)}{(1 - \cos \theta)^2 - (i \sin \theta)^2}$$

$$= \frac{(1 + i \sin \theta)^2 - \cos^2 \theta}{1 - 2 \cos \theta + \cos^2 \theta + \sin^2 \theta}$$

$$= \frac{1 + 2i \sin \theta - \sin^2 \theta - \cos^2 \theta}{1 - 2 \cos \theta + \cos^2 \theta + \sin^2 \theta}$$

$$= \frac{1 + 2i \sin \theta - 1}{1 - 2 \cos \theta + \cos^2 \theta + \sin^2 \theta}$$

$$= \frac{2i \sin \theta}{1 - 2 \cos \theta + \cos^2 \theta + \sin^2 \theta} [\because \cos^2 \theta + \sin^2 \theta = 1]$$

$$= \frac{2i \sin \theta}{1 - 2 \cos \theta + 1}$$

$$= \frac{2i \sin \theta}{2 - 2 \cos \theta}$$

$$= \frac{i \sin \theta}{1 - \cos \theta}$$

$$= \frac{i 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}}$$

$$= \frac{i \cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} = i \cot \frac{\theta}{2}$$

Q16(i)

We have,

$$x = \frac{3-5i}{2}$$

$$\Rightarrow 2x = 3-5i$$

$$\Rightarrow 2x-3 = -5i$$

$$\Rightarrow (2x-3)^2 = (-5i)^2$$

$$\Rightarrow 4x^2+9-12x = -25$$

$$\Rightarrow 4x^2-12x+34 = 0$$

$$\Rightarrow 2(2x^2-6x+17) = 0$$

$$\Rightarrow 2x^2-6x+17 = 0 \quad \dots\dots\dots(i)$$

$$\therefore 2x^3+2x^2-7x+72$$

$$= x(2x^2-6x+17)+6x^2-17x+2x^2-7x+72 \text{ (adding and subtracting } 6x^2 \text{ and } 17x)$$

$$= x \times 0 + 8x^2 - 24x + 72 \quad \text{(using (i))}$$

$$= 4(2x^2-6x+17)+4$$

$$= 4 \times 0 + 4 \quad \text{(using (i))}$$

$$= 4$$

Q16(ii)

We have,

$$x = 3+2i$$

$$\Rightarrow x-3 = 2i$$

$$\Rightarrow (x-3)^2 = (2i)^2$$

$$\Rightarrow x^2+3^2-2 \times 3 \times x = -4$$

$$\Rightarrow x^2+9-6x+4 = 0$$

$$\Rightarrow x^2-6x+13 = 0 \quad \dots\dots\dots(i)$$

Now,

$$x^4 - 4x^3 + 4x^2 + 8x + 44$$

$$= x^2(x^2-6x+13)+6x^2-13x^2-4x^3+4x^2+8x+44 \quad \text{(adding and subtracting } 6x^3 \text{ and } 13x^2)$$

$$= x^2 \times 0 + 2x^3 - 9x^2 + 8x + 44 \quad \text{(using (i))}$$

$$= 2x(x^2-6x+13)+12x^2-26x-9x^2+8x+44 \quad \text{(adding and subtracting } 12x^2 \text{ and } 26x)$$

$$= 2x \times 0 + 3x^2 - 18x + 44 \quad \text{(using (i))}$$

$$= 3(x^2-6x+13)+5$$

$$= 3 \times 0 + 5 \quad \text{(using (i))}$$

$$= 5$$

Q16(iii)

We have,

$$x = -1 + i\sqrt{2}$$

$$\Rightarrow x + 1 = i\sqrt{2}$$

$$\Rightarrow (x + 1)^2 = (i\sqrt{2})^2 \quad \text{(squaring both sides)}$$

$$\Rightarrow x^2 + 1 + 2x = -2$$

$$\Rightarrow x^2 + 2x + 3 = 0 \dots\dots\dots (i)$$

Now,

$$x^4 + 4x^3 + 6x^2 + 4x + 9$$

$$= x^2(x^2 + 2x + 3) + 2x^3 + 3x^2 + 4x + 9$$

$$= x^2 \times 0 + 2x(x^2 + 2x + 3) - x^2 - 2x + 9 \quad \text{(using (i))}$$

$$= 2x \times 0 - (x^2 + 2x + 3) + 3 + 9 \quad \text{(using (i) and adding and subtracting 3)}$$

$$= -0 + 3 + 9 \quad \text{(using (i))}$$

$$= 12$$

Q16(iv)

We have,

$$x = \frac{1+i}{\sqrt{2}}$$

$$\Rightarrow \sqrt{2}x = 1+i$$

$$\Rightarrow (\sqrt{2}x)^2 = (1+i)^2 \quad \text{(squaring both sides)}$$

$$\begin{aligned} \Rightarrow 2x^2 &= 1^2 + (i)^2 + 2 \times 1 \times i \\ &= 1 - 1 + 2i \end{aligned}$$

$$\Rightarrow 2x^2 = 2i$$

$$\Rightarrow x^2 = i$$

$$\Rightarrow (x^2)^2 = (i)^2 \quad \text{(squaring both sides)}$$

$$\Rightarrow x^4 = -1$$

$$\Rightarrow x^4 + 1 = 0 \dots\dots\dots (i)$$

Now

$$x^6 + x^4 + x^2 + 1$$

$$= x^6 + x^2 + x^4 + 1$$

$$= x^2(x^4 + 1) + 1(x^4 + 1)$$

$$= x^2 \times 0 + 1 \times 0 \quad \text{(using (i))}$$

$$= 0$$

Q16(v)

$$x = (-2 - \sqrt{3})$$

$$x^2 = (-2 - \sqrt{3})^2 = 4 + 4\sqrt{3} + 3 = 1 + 4\sqrt{3}$$

$$x^3 = (1 + 4\sqrt{3})(-2 - \sqrt{3}) = -2 - 8\sqrt{3} - \sqrt{3} - 12 = -14 - 9\sqrt{3}$$

$$x^4 = (1 + 4\sqrt{3})^2 = 1 + 8\sqrt{3} + 48 = 49 + 8\sqrt{3}$$

$$\begin{aligned} 2x^4 + 5x^3 + 7x^2 - x + 41 &= 2(-47 + 8\sqrt{3}) + 5(-14 - 9\sqrt{3}) + 7(1 + 4\sqrt{3}) - (-2 - \sqrt{3}) + 41 \\ &= -94 + 16\sqrt{3} + 50 - 45\sqrt{3} + 7 + 28\sqrt{3} + 2 + \sqrt{3} + 41 \\ &= (-94 + 50 + 7 + 2 + 41) + (16\sqrt{3} - 45\sqrt{3} + 28\sqrt{3} + \sqrt{3}) \\ &= 6 + 0 \\ &= 6 \end{aligned}$$

Q17

$$\begin{aligned} &(1-i)^n \left(1 - \frac{1}{i}\right)^n \\ &= (1-i)^n \left(\frac{i-1}{i}\right)^n \\ &= \left\{ \frac{(1-i)(i-1)}{i} \right\}^n \\ &= \left\{ \frac{(1-i)(1-i)}{-i} \right\}^n \\ &= \left\{ \frac{(1-i)^2}{-i} \right\}^n \\ &= \left\{ \frac{1-2i-1}{-i} \right\}^n \\ &= \left\{ \frac{-2i}{-i} \right\}^n = 2^n \end{aligned}$$

Q18

$$(1+i)z = (1-i)\bar{z}$$

$$\Rightarrow z = \frac{(1-i)\bar{z}}{(1+i)}$$

$$\Rightarrow z = \frac{(1-i)(1-i)\bar{z}}{(1+i)(1-i)} \quad [\text{Rationalizing the denominator}]$$

$$\Rightarrow z = \frac{(1-2i-1)\bar{z}}{(1+1)}$$

$$\Rightarrow z = \frac{-2i\bar{z}}{2}$$

$$\Rightarrow z = -i\bar{z}$$

Q19

$$\operatorname{Re}(z^2) = 0, |z|=2$$

$$\text{Let } z = x + iy$$

$$z^2 = 0$$

$$\Rightarrow (x+iy)^2 = 0$$

$$\Rightarrow x^2 + 2ixy - y^2 = 0$$

$$\Rightarrow x^2 - y^2 = 0 \dots (i), \text{ which is the real part of } (x+iy)^2.$$

$$|z|=2$$

$$\Rightarrow \sqrt{x^2 + y^2} = 2$$

$$\Rightarrow x^2 + y^2 = 4 \dots (ii)$$

Adding (i) and (ii), we get

$$2x^2 = 4$$

$$\Rightarrow x^2 = 2$$

$$\Rightarrow x = \pm\sqrt{2}, y = \pm\sqrt{2}$$

$$x+iy = \sqrt{2} + i\sqrt{2}$$

$$= -\sqrt{2} - i\sqrt{2}$$

$$= \sqrt{2} - i\sqrt{2}$$

$$= -\sqrt{2} + i\sqrt{2}$$

Q20

let $z = x + iy$,

$$\frac{z-1}{z+1}$$

$$= \frac{x+iy-1}{x+iy+1}$$

$$= \frac{x-1+iy}{x+1+iy}$$

$$= \frac{(x-1+iy)(x+1-iy)}{(x+1+iy)(x+1-iy)} \text{ [Rationalizing the denominator]}$$

$$= \frac{(x-1+iy)(x+1-iy)}{(x+1)^2 - (iy)^2}$$

$$= \frac{x^2 + x - iy - x - 1 + iy + ix - iy + y^2}{x^2 + 2x + 1 + y^2}$$

$$= \frac{x^2 - 1 + 2iy + y^2}{x^2 + 2x + 1 + y^2}$$

$$= \frac{x^2 + y^2 - 1}{x^2 + 2x + 1 + y^2} + i \frac{2y}{x^2 + 2x + 1 + y^2}$$

\therefore It is a purely imaginary no. therefore real part = 0

$$\frac{x^2 + y^2 - 1}{x^2 + 2x + 1 + y^2} = 0$$

$$\Rightarrow x^2 + y^2 - 1 = 0$$

$$\Rightarrow x^2 + y^2 = 1$$

$$\Rightarrow \sqrt{x^2 + y^2} = 1$$

$$\Rightarrow |z| = 1$$

Q21

$$\text{Let } z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$$

$$|z_1| = 1 \Rightarrow x_1^2 + y_1^2 = 1$$

$$z_2 = \frac{z_1 - 1}{z_1 + 1}$$

$$x_2 + iy_2 = \frac{x_1 + iy_1 - 1}{x_1 + iy_1 + 1}$$

$$\Rightarrow x_2 + iy_2 = \frac{x_1 - 1 + iy_1}{x_1 + 1 + iy_1}$$

$$\Rightarrow x_2 + iy_2 = \frac{(x_1 - 1 + iy_1)(x_1 + 1 - iy_1)}{(x_1 + 1 + iy_1)(x_1 + 1 - iy_1)} \text{ [Rationalizing the denominator]}$$

$$\Rightarrow x_2 + iy_2 = \frac{(x_1 - 1)(x_1 + 1) - iy_1(x_1 - 1) + iy_1(x_1 + 1) + y_1^2}{(x_1 + 1)^2 - (iy_1)^2}$$

$$\Rightarrow x_2 + iy_2 = \frac{x_1^2 - 1 + y_1^2 - iy_1x_1 + iy_1x_1 + y_1^2 + y_1^2}{(x_1 + 1)^2 - (iy_1)^2}$$

$$\Rightarrow x_2 + iy_2 = \frac{x_1^2 + y_1^2 - 1 + 2iy_1}{(x_1 + 1)^2 - (iy_1)^2}$$

$$\Rightarrow x_2 + iy_2 = \frac{1 - 1 + 2iy_1}{(x_1 + 1)^2 - (iy_1)^2} [\because x_1^2 + y_1^2 = 1]$$

$$\Rightarrow x_2 + iy_2 = \frac{2iy_1}{(x_1 + 1)^2 - (iy_1)^2} [\because x_1^2 + y_1^2 = 1]$$

Since there is no real part in the RHS, therefore $x_2 = 0$.

The real part of the $z_2 = 0$.

Q22

$$\text{Let } z = x + iy$$

$$|z+1| = z + 2(1+i)$$

$$\Rightarrow |x+iy+1| = x+iy+2+2i$$

$$\Rightarrow \sqrt{(x+1)^2 + y^2} = (x+2) + i(y+2)$$

Comparing real and imaginary parts, we get

$$x+2 = \sqrt{x^2 + 2x+1+y^2} \quad \text{and } y+2=0$$

$$y+2=0$$

$$\Rightarrow y = -2$$

$$\& (x+2)^2 = x^2 + 2x+1+y^2$$

$$\Rightarrow x^2 + 4x+4 = x^2 + 2x+1+y^2$$

$$\Rightarrow 2x+3 = y^2$$

$$\Rightarrow 2x+3 = (-2)^2$$

$$\Rightarrow 2x+3 = 4$$

$$\Rightarrow 2x = 1$$

$$\Rightarrow x = \frac{1}{2}$$

$$\therefore z = x+iy = \frac{1}{2} - i2$$

Q23

$$\text{Let } z = x + iy$$

$$|z| = z + 1 + 2i$$

$$\Rightarrow |x + iy| = x + iy + 1 + 2i$$

$$\Rightarrow \sqrt{x^2 + y^2} = (x+1) + i(y+2)$$

$$\Rightarrow x^2 + y^2 = (x+1)^2 + 2i(x+1)(y+2) - (y+2)^2 \quad [\text{Squaring both sides}]$$

$$\Rightarrow x^2 + y^2 = x^2 + 2x + 1 + 2i(xy + 2x + y + 2) - (y^2 + 4y + 4)$$

$$\Rightarrow 2y^2 - 2x + 4y + 4 = 2i(xy + 2x + y + 2)$$

$$\Rightarrow y^2 - x + 2y + 2 = i(xy + 2x + y + 2)$$

$$\Rightarrow (y^2 - x + 2y + 2) - i(xy + 2x + y + 2) = 0$$

Comparing we get,

$$(xy + 2x + y + 2) = 0$$

$$\Rightarrow (x+1)(y+2) = 0$$

$$\Rightarrow x = -1 \text{ \& } y = -2$$

$$\text{Also, } (y^2 - x + 2y + 2) = 0$$

$$\text{Taking } x = -1, (y^2 - (-1) + 2y + 2) = 0$$

$$\Rightarrow (y^2 + 2y + 3) = 0$$

Does not have a solution since roots will be imaginary

$$\text{Taking } y = -2, (4 - x - 4 + 2) = 0$$

$$\Rightarrow x = 2$$

$$\therefore z = x + iy = 2 - 2i$$

Q24

$$\begin{aligned}
(1+i)^{2n} &= (1-i)^{2n} \\
\Rightarrow \left(\frac{1+i}{1-i}\right)^{2n} &= 1 \\
\Rightarrow \left(\frac{(1+i)(1+i)}{(1-i)(1+i)}\right)^{2n} &= 1 \quad [\text{Rationalizing the denominator}] \\
\Rightarrow \left(\frac{1+2i-1}{1+1}\right)^{2n} &= 1 \\
\Rightarrow \left(\frac{2i}{2}\right)^{2n} &= 1 \\
\Rightarrow i^{2n} &= 1 \\
\therefore n &= 2
\end{aligned}$$

Q25

$$\begin{aligned}
|z_1 + z_2 + z_3| &= \left| \frac{z_1 \bar{z}_1}{z_1} + \frac{z_2 \bar{z}_2}{z_2} + \frac{z_3 \bar{z}_3}{z_3} \right| \\
&= \left| \frac{|z_1|^2}{z_1} + \frac{|z_2|^2}{z_2} + \frac{|z_3|^2}{z_3} \right| \\
&= \left| \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right| \dots \dots \dots [\because |z_1| = |z_2| = |z_3| = 1] \\
&= \left| \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right| \\
&= 1
\end{aligned}$$

Q26

Let $z = x + iy$

$$z^2 = (x + iy)^2 = x^2 - y^2 + 2xyi$$

$$|z|^2 = z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$$

$$z^2 + |z|^2 = 0$$

$$x^2 - y^2 + 2xyi + x^2 + y^2 = 0$$

$$2x^2 + 2xyi = 0$$

$$\Rightarrow 2x^2 = 0 \text{ and } 2xy = 0$$

$$\Rightarrow x = 0 \text{ and } y \in \mathbb{R}$$

$$\therefore z = 0 + iy \text{ where } y \in \mathbb{R}$$

Ex 13.3

Q1(i)

$$\text{Let } z = -5 + 12i$$

$$\begin{aligned}\Rightarrow |z| &= \sqrt{(-5)^2 + 12^2} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13\end{aligned}$$

$$\begin{aligned}\therefore \sqrt{-5 + 12i} &= \pm \left\{ \sqrt{\frac{13 + (-5)}{2}} + i \sqrt{\frac{13 - (-5)}{2}} \right\} && (\because y > 0) \\ &= \pm \left\{ \sqrt{\frac{8}{2}} + i \sqrt{\frac{18}{2}} \right\} \\ &= \pm \{2 + 3i\}\end{aligned}$$

Q1(ii)

$$\text{let } z = -7 - 24i$$

$$\begin{aligned}\text{then } |z| &= \sqrt{(-7)^2 + (-24)^2} \\ &= \sqrt{49 + 576} \\ &= \sqrt{625} \\ &= 25\end{aligned}$$

$$\begin{aligned}\therefore \sqrt{-7 - 24i} &= \pm \left\{ \sqrt{\frac{25 - 7}{2}} - i \sqrt{\frac{25 + 7}{2}} \right\} && (\because y < 0) \\ &= \pm \left\{ \sqrt{\frac{18}{2}} - i \sqrt{\frac{32}{2}} \right\} \\ &= \pm \{ \sqrt{9} - i \sqrt{16} \} \\ &= \pm \{3 - 4i\}\end{aligned}$$

Q1(iii)

$$\text{let } z = 1 - i$$

$$\begin{aligned}\text{then } |z| &= \sqrt{1^2 + (-1)^2} \\ &= \sqrt{1+1} \\ &= \sqrt{2}\end{aligned}$$

$$\begin{aligned}\therefore \sqrt{1-i} &= \pm \left(\sqrt{\frac{\sqrt{2}+1}{2}} - i \sqrt{\frac{\sqrt{2}-1}{2}} \right) \quad (\because y < 0) \\ &= \pm \left(\sqrt{\frac{\sqrt{2}+1}{2}} - i \sqrt{\frac{\sqrt{2}-1}{2}} \right)\end{aligned}$$

Q1(iv)

$$\text{let } z = -8 - 6i$$

$$\begin{aligned}\text{then } |z| &= \sqrt{(-8)^2 + (-6)^2} \\ &= \sqrt{64+36} \\ &= \sqrt{100} \\ &= 10\end{aligned}$$

$$\begin{aligned}\therefore \sqrt{-8-6i} &= \pm \left\{ \sqrt{\frac{10-8}{2}} - i \sqrt{\frac{10+8}{2}} \right\} \quad (\because y < 0) \\ &= \pm \left\{ \sqrt{\frac{2}{2}} - i \sqrt{\frac{18}{2}} \right\} \\ &= \pm \left\{ \sqrt{1} - i \sqrt{9} \right\} \\ &= \pm \{1 - 3i\}\end{aligned}$$

Q1(v)

$$\text{let } z = 8 - 15i$$

$$\begin{aligned}\text{then } |z| &= \sqrt{(8)^2 + (-15)^2} \\ &= \sqrt{64+225} \\ &= \sqrt{289} \\ &= 17\end{aligned}$$

$$\begin{aligned}\therefore \sqrt{8-15i} &= \pm \left\{ \sqrt{\frac{17+8}{2}} - i \sqrt{\frac{17-8}{2}} \right\} \quad (\because y < 0) \\ &= \pm \left\{ \sqrt{\frac{25}{2}} - i \sqrt{\frac{9}{2}} \right\} \\ &= \pm \left\{ \frac{5}{\sqrt{2}} - i \frac{3}{\sqrt{2}} \right\} \\ &= \pm \frac{1}{\sqrt{2}} \{5 - 3i\}\end{aligned}$$

Q1(vi)

$$\text{Let } z = -11 - 60\sqrt{-1}$$

$$\Rightarrow z = -11 - 60i \quad (\because \sqrt{-1} = i)$$

$$\begin{aligned} \text{Then } |z| &= \sqrt{(-11)^2 + (-60)^2} \\ &= \sqrt{121 + 3600} \\ &= \sqrt{3721} \\ &= 61 \end{aligned}$$

$$\begin{aligned} \therefore \sqrt{-11 - 60i} &= \pm \left\{ \sqrt{\frac{61-11}{2}} - i\sqrt{\frac{61+11}{2}} \right\} \quad (\because y < 0) \\ &= \pm \left\{ \sqrt{\frac{50}{2}} - i\sqrt{\frac{72}{2}} \right\} \\ &= \pm \left\{ \sqrt{25} - i\sqrt{36} \right\} \\ &= \pm \{5 - 6i\} \end{aligned}$$

Q1(vii)

$$\text{let } z = 1 + 4\sqrt{-3}$$

$$= 1 + 4\sqrt{3} \times \sqrt{-1} \quad (\because \sqrt{-3} = \sqrt{3} \times \sqrt{-1})$$

$$\Rightarrow z = 1 + 4\sqrt{3}i$$

$$\begin{aligned} \therefore |z| &= \sqrt{(1)^2 + (4\sqrt{3})^2} \\ &= \sqrt{1 + 48} \\ &= \sqrt{49} \\ &= 7 \end{aligned}$$

$$\begin{aligned} \text{Hence } \sqrt{1 + 4\sqrt{-3}} &= \pm \left\{ \sqrt{\frac{7+1}{2}} + i\sqrt{\frac{7-1}{2}} \right\} \quad (\because y > 0) \\ &= \pm \left\{ \sqrt{\frac{8}{2}} + i\sqrt{\frac{6}{2}} \right\} \\ &= \pm \left\{ \sqrt{4} + i\sqrt{3} \right\} \\ &= \pm \{2 + \sqrt{3}i\} \end{aligned}$$

Q1(viii)

$$\text{let } z = 4i$$

$$\begin{aligned}\text{then } |z| &= |4i| \\ &= 4|i| \\ &= 4\end{aligned}$$

$$\begin{aligned}(\because |z_1 z_2| &= |z_1| \times |z_2|) \\ (\because |i| &= 1)\end{aligned}$$

$$\begin{aligned}\therefore \sqrt{4i} &= \pm \left\{ \sqrt{\frac{4+0}{2}} + i \sqrt{\frac{4-0}{2}} \right\} \\ &= \pm \left\{ \sqrt{2} + i\sqrt{2} \right\} \\ &= \pm \sqrt{2} (1+i)\end{aligned}$$

$$(\because y > 0)$$

Q1(ix)

$$\text{let } z = -i$$

$$\begin{aligned}\text{then } |z| &= |-i| \\ &= |-1| \times |i| \\ &= 1 \times i \\ &= 1\end{aligned}$$

$$\begin{aligned}(\because |z_1 z_2| &= |z_1| \times |z_2|) \\ (\because |i| &= 1)\end{aligned}$$

$$\begin{aligned}\therefore \sqrt{-i} &= \pm \left\{ \sqrt{\frac{1+0}{2}} - i \sqrt{\frac{1-0}{2}} \right\} \\ &= \pm \left\{ \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right\} \\ &= \pm \frac{1}{\sqrt{2}} (1-i)\end{aligned}$$

$$(\because y < 0)$$

Ex 13.4

Q1(i)

The polar form of a complex number $z = x + iy$, is given by $z = |z|(\cos \theta + i \sin \theta)$

where,

$$|z| = \sqrt{x^2 + y^2} \text{ and}$$

$$\arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

let $z = 1 + i$

$$\begin{aligned} |z| &= \sqrt{1^2 + 1^2} \\ &= \sqrt{2} \end{aligned}$$

$\therefore x, y > 0$, so θ lies in first quadrant

Now,

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{b}{a}\right) \\ &= \tan^{-1}\left(\frac{1}{1}\right) && [\because a = 1 \text{ and } b = 1] \\ &= \tan^{-1}(1) \\ &= \tan^{-1}\left(\frac{\tan \pi}{4}\right) && \left(\because \frac{\tan \pi}{4} = 1\right) \\ &= \frac{\pi}{4} && (\because \tan^{-1}(\tan x) = x) \end{aligned}$$

$$\Rightarrow \arg(z) = \frac{\pi}{4}$$

Polar form of $1 + i$ is given by $z = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$

Q1(ii)

The polar form of a complex number $z = x + iy$, is given by $z = |z|(\cos \theta + i \sin \theta)$

where,

$$|z| = \sqrt{x^2 + y^2} \text{ and}$$

$$\arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\text{let } z = \sqrt{3} + i$$

$$\begin{aligned} |z| &= \sqrt{(\sqrt{3})^2 + (1)^2} \\ &= \sqrt{3+1} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

$$\because x = \sqrt{3} > 0 \text{ \& } y = 1 > 0,$$

$\therefore \theta$ lies in first quadrant

Hence

$$\begin{aligned} \theta = \arg(z) &= \tan^{-1}\left(\frac{y}{x}\right) \\ &= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \\ &= \tan^{-1}\left(\frac{\tan \frac{\pi}{6}}{1}\right) \\ &= \tan^{-1}\left(\because \tan^{-1}(\tan x) = x\right) \end{aligned}$$

polar form is given by $z = |z|(\cos \theta + i \sin \theta)$

$$\text{i.e. } z = 2\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$$

Q1(iii)

$$\text{Modulus, } |1-i| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\text{Argument, } \arg(1-i) = \tan^{-1}\left(\frac{-1}{1}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$$

$$\text{Polar form, } \sqrt{2}\left(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4}\right)$$

Q1(iv)

$$\frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)} = \frac{(1-i)^2}{1^2-i^2} = \frac{1-2i-1}{1+1} = \frac{-2i}{2} = -i$$

$$\text{Modulus, } \left| \frac{1-i}{1+i} \right| = |-i| = 1$$

$$\text{Argument, } \tan^{-1} \left(\frac{-1}{0} \right) = -\frac{\pi}{2}$$

$$\text{Polar Form, } z = r(\cos \theta + i \sin \theta)$$

$$z = \left(\cos \frac{\pi}{2} - i \sin \frac{\pi}{2} \right)$$

Q1(v)

$$\text{Modulus, } \left| \frac{1}{1+i} \right|$$

$$= \left| \frac{1(1-i)}{(1+i)(1-i)} \right| \text{ [Rationalizing the denominator]}$$

$$= \left| \frac{1-i}{1^2-i^2} \right| = \left| \frac{1-i}{2} \right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\text{Argument, } \tan^{-1}(-1) = -\frac{\pi}{4}$$

$$\text{Polar Form} = \cos \left(\frac{\pi}{4} \right) - i \sin \left(\frac{\pi}{4} \right)$$

Q1(vi)

The polar form of a complex number $z = x + iy$, is given by $z = |z|(\cos \theta + i \sin \theta)$ where,

$$|z| = \sqrt{x^2 + y^2} \text{ and}$$

$$\arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\begin{aligned} \text{let } z &= \frac{1+2i}{1-3i} \\ &= \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i} \\ &= \frac{1(1+3i) + 2i(1+3i)}{1^2 + 3^2} \\ &= \frac{1+3i+2i-6}{1+9} \\ &= \frac{-5+5i}{10} \\ &= \frac{-5}{10} + \frac{5}{10}i \\ &= \frac{-1}{2} + \frac{1}{2}i \end{aligned}$$

$$\begin{aligned} \therefore |z| &= \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \\ &= \sqrt{\frac{1}{4} + \frac{1}{4}} \\ &= \sqrt{\frac{2}{4}} \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

Here $x = \frac{-1}{2} < 0$ & $y = \frac{1}{2} > 0$, $\therefore \theta$ lies in quadrant II

$$\begin{aligned} \theta = \arg(z) &= \tan^{-1} \frac{\frac{1}{2}}{\frac{-1}{2}} \\ &= \tan^{-1}(-1) \\ &= \tan^{-1}\left(-\tan \frac{\pi}{4}\right) \\ &= \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{4}\right)\right) \quad (\because \tan(\pi - \theta) = -\tan \theta) \\ &= \pi - \frac{\pi}{4} \\ &= \frac{3\pi}{4} \end{aligned}$$

The polar form is given by $z = \frac{1}{\sqrt{2}} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$

Q1(vii)

The polar form of a complex number $z = x + iy$, is given by $z = |z|(\cos \theta + i \sin \theta)$

where,

$$|z| = \sqrt{x^2 + y^2} \text{ and}$$

$$\arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\text{let } z = \sin 120^\circ - i \cos 120^\circ$$

$$= \sin\left(\frac{\pi}{2} + \frac{\pi}{6}\right) - i \cos\left(\frac{\pi}{2} + \frac{\pi}{6}\right) \quad \left(\because 120^\circ = \frac{\pi}{2} + \frac{\pi}{6}\right)$$

$$\Rightarrow z = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \quad \left(\because \sin\left(\frac{\pi}{2} + \theta\right) = \cos \theta \text{ \& } \cos\left(\frac{\pi}{2} + \theta\right) = -\sin \theta\right)$$

Here z is already in polar form

$$\text{with } |z| = 1 \text{ \& } \theta = \arg(z) = \frac{\pi}{6}$$

Q1(viii)

The polar form of a complex number $z = x + iy$, is given by $z = |z|(\cos \theta + i \sin \theta)$

where,

$$|z| = \sqrt{x^2 + y^2} \text{ and}$$

$$\arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\begin{aligned} \text{let } z &= \frac{-16}{1+i\sqrt{3}} \\ &= \frac{-16}{1+i\sqrt{3}} \times \frac{1-i\sqrt{3}}{1-i\sqrt{3}} \\ &= \frac{-16(1-i\sqrt{3})}{(1)^2 + (\sqrt{3})^2} \\ &= \frac{-16(1-i\sqrt{3})}{1+3} \\ &= \frac{-16}{4}(1-i\sqrt{3}) \\ &= -4(1-i\sqrt{3}) \\ &= -4 + 4\sqrt{3}i \end{aligned}$$

$$\begin{aligned} \therefore |z| &= \sqrt{(-4)^2 + (4\sqrt{3})^2} \\ &= \sqrt{16 + 48} \\ &= \sqrt{64} \\ &= 8 \end{aligned}$$

Here $x = -4 < 0$ & $y = 4\sqrt{3} > 0$, $\therefore \theta$ lies in quadrant II

$$\begin{aligned} \theta = \arg(z) &= \tan^{-1}\left(\frac{4\sqrt{3}}{-4}\right) \\ &= \tan^{-1}(-\sqrt{3}) \\ &= \tan^{-1}\left(-\tan\frac{\pi}{3}\right) \\ &= \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{3}\right)\right) \quad (\because \tan(\pi - \theta) = -\tan\theta) \\ &= \pi - \frac{\pi}{3} \\ &= \frac{2\pi}{3} \end{aligned}$$

The polar form is given by $z = 8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

Q2

$$z = (i^{25})^3 = (i)^3 = -i$$

$$|z| = 1,$$

$$\arg(z) = \tan^{-1}\left(\frac{-1}{0}\right) = -\frac{\pi}{2}$$

$$\text{Polar Form: } \cos\left(\frac{\pi}{2}\right) - i \sin\left(\frac{\pi}{2}\right) = \cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right)$$

Q3(i)

Let $z = 1 + i \tan \alpha$.

$\tan \alpha$ is periodic function with period π

So, let us take α lying in the interval $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.

Case - I : When $\alpha \in \left[0, \frac{\pi}{2}\right)$

$$|z| = \sqrt{1 + \tan^2 \alpha} = \sqrt{\sec^2 \alpha} = |\sec \alpha| = \sec \alpha$$

Let β be acute angle given by $\tan \beta = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$.

$$\tan \beta = |\tan \alpha| = \tan \alpha$$

$$\Rightarrow \beta = \alpha$$

As z is represented by a point in first quadrant.

$$\therefore \arg(z) = \beta = \alpha$$

So polar form of z is $\sec \alpha (\cos \alpha + i \sin \alpha)$

Case - II : When $\alpha \in \left(\frac{\pi}{2}, \pi\right]$

$$|z| = \sqrt{1 + \tan^2 \alpha} = \sqrt{\sec^2 \alpha} = |\sec \alpha| = -\sec \alpha$$

Let β be acute angle given by $\tan \beta = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$.

$$\tan \beta = |\tan \alpha| = -\tan \alpha = \tan(\pi - \alpha)$$

$$\Rightarrow \beta = \pi - \alpha$$

As z is represented by a point in fourth quadrant.

$$\therefore \arg(z) = -\beta = \alpha - \pi$$

So polar form of z is $-\sec \alpha (\cos(\alpha - \pi) + i \sin(\alpha - \pi))$.

Q3(ii)

Let $z = \tan \alpha - i$

$\tan \alpha$ is periodic function with period π

So, let us take α lying in the interval $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.

Case - I : When $\alpha \in \left[0, \frac{\pi}{2}\right)$

$$|z| = \sqrt{\tan^2 \alpha + 1} = \sqrt{\sec^2 \alpha} = |\sec \alpha| = \sec \alpha$$

Let β be acute angle given by $\tan \beta = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$.

$$\tan \beta = \frac{1}{|\tan \alpha|} = |\cot \alpha| = \cot \alpha = \tan\left(\frac{\pi}{2} - \alpha\right)$$

$$\Rightarrow \beta = \frac{\pi}{2} - \alpha$$

As z is represented by a point in fourth quadrant.

$$\therefore \arg(z) = -\beta = \alpha - \frac{\pi}{2}.$$

So polar form of z is $\sec \alpha \left(\cos\left(\alpha - \frac{\pi}{2}\right) + i \sin\left(\alpha - \frac{\pi}{2}\right) \right)$

Case - II : When $\alpha \in \left(\frac{\pi}{2}, \pi\right]$

$$|z| = \sqrt{\tan^2 \alpha + 1} = \sqrt{\sec^2 \alpha} = |\sec \alpha| = -\sec \alpha$$

Let β be acute angle given by $\tan \beta = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$.

$$\tan \beta = \frac{1}{|\tan \alpha|} = |\cot \alpha| = -\cot \alpha = \tan\left(\alpha - \frac{\pi}{2}\right)$$

$$\Rightarrow \beta = \alpha - \frac{\pi}{2}$$

As z is represented by a point in third quadrant.

$$\therefore \arg(z) = \pi + \beta = \frac{\pi}{2} + \alpha.$$

So polar form of z is $-\sec \alpha \left(\cos\left(\frac{\pi}{2} + \alpha\right) + i \sin\left(\frac{\pi}{2} + \alpha\right) \right)$.

Q3(iii)

$$\text{Let } z = (1 - \sin \alpha) + i \cos \alpha$$

Since sine and cosine are periodic functions with period 2π

So, let us take α lying in the interval $[0, 2\pi]$.

$$\text{Now, } z = (1 - \sin \alpha) + i \cos \alpha$$

$$\Rightarrow |z| = \sqrt{(1 - \sin \alpha)^2 + \cos^2 \alpha} = \sqrt{2 - 2\sin \alpha} = \sqrt{2}\sqrt{1 - \sin \alpha}$$

$$\Rightarrow |z| = \sqrt{2} \sqrt{\left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2}\right)^2} = \sqrt{2} \left| \cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right|$$

$$\text{Let } \beta \text{ be acute angle given by } \tan \beta = \frac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$$

$$\tan \beta = \frac{|\cos \alpha|}{|1 - \sin \alpha|} = \frac{|\cos \alpha|}{|1 - \sin \alpha|} = \frac{\left| \cos^2 \frac{\alpha}{2} - \sin^2 \frac{\alpha}{2} \right|}{\left| \left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right)^2 \right|} = \frac{\left| \cos \frac{\alpha}{2} + \sin \frac{\alpha}{2} \right|}{\left| \cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right|}$$

$$\Rightarrow \tan \beta = \frac{\left| 1 + \tan \frac{\alpha}{2} \right|}{\left| 1 - \tan \frac{\alpha}{2} \right|} = \left| \tan \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) \right|$$

Following cases arise:

Case I: When $0 \leq \alpha < \frac{\pi}{2}$

$$\cos \frac{\alpha}{2} > \sin \frac{\alpha}{2} \text{ and } \frac{\pi}{4} + \frac{\alpha}{2} \in \left[\frac{\pi}{4}, \frac{\pi}{2} \right)$$

$$\therefore |z| = \sqrt{2} \left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right)$$

$$\text{and, } \tan \beta = \left| \tan \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) \right| = \tan \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) \Rightarrow \beta = \frac{\pi}{4} + \frac{\alpha}{2}$$

Clearly, z lies in the first quadrant.

$$\therefore \arg(z) = \frac{\pi}{4} + \frac{\alpha}{2}$$

$$\text{So polar form of } z \text{ is } \sqrt{2} \left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right) \left(\cos \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) + i \sin \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) \right)$$

Case II: When $\frac{\pi}{2} < \alpha < \frac{3\pi}{2}$

$$\cos \frac{\alpha}{2} < \sin \frac{\alpha}{2} \text{ and } \frac{\pi}{4} + \frac{\alpha}{2} \in \left(\frac{\pi}{2}, \pi \right)$$

$$\therefore |z| = \sqrt{2} \left| \cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right| = -\sqrt{2} \left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right)$$

$$\text{and, } \tan \beta = \left| \tan \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) \right| = -\tan \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) = \tan \left\{ \pi - \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) \right\} = \tan \left(\frac{3\pi}{4} - \frac{\alpha}{2} \right)$$

$$\Rightarrow \beta = \frac{3\pi}{4} - \frac{\alpha}{2}$$

Since $1 - \sin \alpha > 0$ and $\cos \alpha < 0$.

Clearly, z lies in the fourth quadrant.

$$\therefore \arg(z) = -\beta = \frac{\alpha}{2} - \frac{3\pi}{4}$$

$$\text{So polar form of } z \text{ is } -\sqrt{2} \left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right) \left(\cos \left(\frac{\alpha}{2} - \frac{3\pi}{4} \right) + i \sin \left(\frac{\alpha}{2} - \frac{3\pi}{4} \right) \right)$$

Case III: When $\frac{3\pi}{2} < \alpha < 2\pi$

$$\cos \frac{\alpha}{2} < \sin \frac{\alpha}{2} \text{ and } \frac{\pi}{4} + \frac{\alpha}{2} \in \left(\pi, \frac{5\pi}{4} \right)$$

$$\therefore |z| = \sqrt{2} \left| \cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right| = -\sqrt{2} \left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right)$$

$$\text{and, } \tan \beta = \left| \tan \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) \right| = \tan \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) = -\tan \left\{ \pi - \left(\frac{\pi}{4} + \frac{\alpha}{2} \right) \right\} = \tan \left(\frac{\alpha}{2} - \frac{3\pi}{4} \right)$$

$$\Rightarrow \beta = \frac{\alpha}{2} - \frac{3\pi}{4}$$

Clearly, $\operatorname{Re}(z) < 0$ and $\operatorname{Im}(z) > 0$.

So, z lies in the first quadrant.

$$\therefore \arg(z) = \beta = \frac{\alpha}{2} - \frac{3\pi}{4}$$

$$\text{So polar form of } z \text{ is } -\sqrt{2} \left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right) \left(\cos \left(\frac{\alpha}{2} - \frac{3\pi}{4} \right) + i \sin \left(\frac{\alpha}{2} - \frac{3\pi}{4} \right) \right).$$

Q3(iv)

$$\text{Let } z = \frac{1-i}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}} = \frac{1-i}{\frac{1}{2} + i \frac{\sqrt{3}}{2}} = \frac{2-2i}{1+i\sqrt{3}} = \frac{(2-2i)(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} = \frac{(2-2\sqrt{3})-i(2\sqrt{3}+2)}{4} = \frac{(1-\sqrt{3})}{2} - i \frac{(\sqrt{3}+1)}{2}$$

$$|z| = \sqrt{\frac{(1-\sqrt{3})^2}{4} + \frac{(\sqrt{3}+1)^2}{4}} = \sqrt{\frac{8}{4}} = \sqrt{2}$$

$$\text{Let } \beta \text{ be acute angle given by } \tan \beta = \frac{|\text{Im}(z)|}{|\text{Re}(z)|}$$

$$\tan \beta = \frac{\left| \frac{-(\sqrt{3}+1)}{2} \right|}{\left| \frac{(1-\sqrt{3})}{2} \right|} = \frac{|-(\sqrt{3}+1)|}{|(1-\sqrt{3})|} = |2+\sqrt{3}| = \tan\left(\frac{7\pi}{12}\right)$$

$$\Rightarrow \beta = \frac{7\pi}{12}$$

Z is represented by a point in second quadrant.

So polar form of z is $\sqrt{2} \left(\cos \frac{7\pi}{12} - i \sin \frac{7\pi}{12} \right)$.

$$|z_1| = |z_2|$$

$$\text{Let } \arg(z_1) = \theta$$

$$\therefore \arg(z_2) = \pi - \theta$$

In polar form, $z_1 = |z_1|(\cos \theta + i \sin \theta)$(i)

$$z_2 = |z_2|(\cos(\pi - \theta) + i \sin(\pi - \theta))$$

$$= |z_2|(-\cos \theta + i \sin \theta)$$

$$= -|z_2|(\cos \theta - i \sin \theta)$$

Finding conjugate of

$$\overline{z_2} = -|z_2|(\cos \theta + i \sin \theta)$$
.....(ii)

(i)/(ii) is equal to

$$\frac{z_1}{z_2} = \frac{|z_1|(\cos \theta + i \sin \theta)}{|z_2|(\cos \theta + i \sin \theta)}$$

$$\frac{z_1}{z_2} = -\frac{|z_1|}{|z_1|} \quad [\because |z_1| = |z_2|]$$

$$\frac{z_1}{z_2} = -1$$

$$z_1 = -z_2$$

$$z_1 = -z_2$$

Hence Proved.

Q4

z_1, z_2 are conjugates implies $z_2 = \overline{z_1}$

z_3, z_4 are conjugates implies $z_4 = \overline{z_3}$

Also we know that $\arg(z_1) + \arg(\overline{z_1}) = 0$

$$\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right)$$

$$= \arg(z_1) - \arg(z_4) + \arg(z_2) - \arg(z_3) \quad [\because \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)]$$

$$= \arg(z_1) - \arg(\overline{z_3}) + \arg(\overline{z_1}) - \arg(z_3)$$

$$= \arg(z_1) + \arg(\overline{z_1}) - \arg(\overline{z_3}) - \arg(z_3)$$

$$= \arg(z_1) + \arg(\overline{z_1}) - [\arg(\overline{z_3}) + \arg(z_3)] [\because \arg(z_1) + \arg(\overline{z_1}) = 0]$$

$$= 0 + 0 = 0$$

Q5

$$\sin \frac{\pi}{5} + i \left(1 - \cos \frac{\pi}{5} \right)$$

$$= 2 \sin \frac{\pi}{10} \cos \frac{\pi}{10} + i 2 \sin^2 \frac{\pi}{10} [\text{Using } \sin 2\theta = 2 \sin \theta \cos \theta \text{ \& } 1 - \cos 2\theta = 2 \sin^2 \theta]$$

$$= 2 \sin \frac{\pi}{10} \left(\cos \frac{\pi}{10} + i \sin \frac{\pi}{10} \right)$$