

8

Mean Value Theorems, Maxima and Minima

8.1 ROLLE'S THEOREM

If a function $f(x)$ is

- (i) continuous in the closed interval $[a, b]$
- (ii) derivable in the open interval (a, b) and
- (iii) $f(a) = f(b)$,

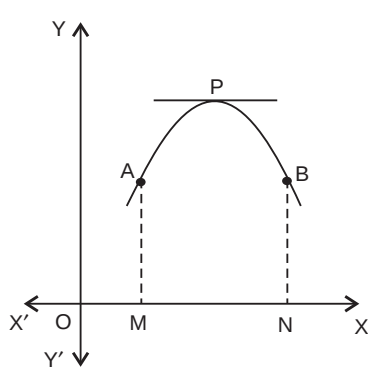
then there exists atleast one real number c in (a, b) such that $f'(c) = 0$.

(We accept it without proof.)

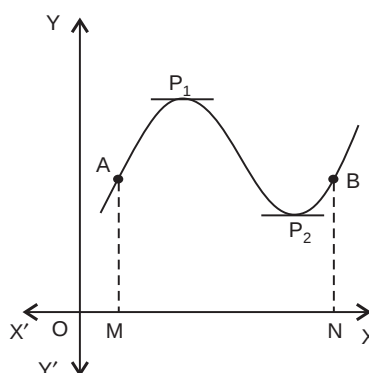
Geometrical Interpretation

Let A, B be the points on the curve $y = f(x)$ corresponding to the real numbers a, b respectively.

Since $f(x)$ is continuous in $[a, b]$, the graph of the curve $y = f(x)$ is continuous from A to B . Again, as $f(x)$ is derivable in (a, b) , the curve $y = f(x)$ has a tangent at each point between A and B . Also as $f(a) = f(b)$ the ordinates of the points A and B are equal i.e. $MA = NB$ (see fig. 8.1).



(i)



(ii)

Fig. 8.1.

Then Rolle's theorem asserts that there is atleast one point lying between A and B such that the tangent at which is parallel to x -axis i.e. there exists atleast one real number c in (a, b) such that $f'(c) = 0$.

There may exist more than one point between A and B the tangents at which are parallel to x -axis (as shown in fig. 8.1 (ii)) i.e. there may exist more than one real number c in (a, b)

such that $f'(c) = 0$. Rolle's theorem ensures the existence of atleast one real number c in (a, b) such that $f'(c) = 0$.

Remarks

1. Rolle's theorem fails for the function which does not satisfy even one of the three conditions.
2. The converse of Rolle's theorem may not be true, for, $f'(x)$ may be zero at a point in (a, b) without satisfying all the three conditions of Rolle's theorem.

ILLUSTRATIVE EXAMPLES

Example 1. Verify Rolle's theorem for the following functions :

(a) $f(x) = x^2 + x - 6$ in $[-3, 2]$

(b) $f(x) = (x^2 - 1)(x - 2)$ in $[-1, 2]$.

Solution. (a) Given $f(x) = x^2 + x - 6$... (1)

- (i) As $f(x)$ is a polynomial function, it is continuous in $[-3, 2]$,
- (ii) $f(x)$ being a polynomial function is derivable in $(-3, 2)$ and
- (iii) $f(-3) = (-3)^2 - 3 - 6 = 0$, $f(2) = 2^2 + 2 - 6 = 0 \Rightarrow f(-3) = f(2)$.

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number c in $(-3, 2)$ such that $f'(c) = 0$.

Differentiating (1) w.r.t. x , we get $f'(x) = 2x + 1$.

$$\text{Now } f'(c) = 0 \Rightarrow 2c + 1 = 0 \Rightarrow c = -\frac{1}{2}.$$

$$\text{So there exists } -\frac{1}{2} \in (-3, 2) \text{ such that } f'\left(-\frac{1}{2}\right) = 0.$$

Hence, Rolle's theorem is verified.

(b) Given $f(x) = (x^2 - 1)(x - 2)$... (1)

- (i) Since $f(x)$ is a polynomial function, it is continuous in $[-1, 2]$,
- (ii) $f(x)$ being a polynomial function is derivable in $(-1, 2)$ and
- (iii) $f(-1) = (1 - 1)(1 - 2) = 0$, $f(2) = (4 - 1)(2 - 2) = 0 \Rightarrow f(-1) = f(2)$.

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number c in $(-1, 2)$ such that $f'(c) = 0$.

Differentiating (1) w.r.t. x , we get

$$f'(x) = (x^2 - 1).1 + (x - 2).2x = 3x^2 - 4x - 1.$$

$$\text{Now } f'(c) = 0 \Rightarrow 3c^2 - 4c - 1 = 0$$

$$\Rightarrow c = \frac{4 \pm \sqrt{16 - 4.3.(-1)}}{2.3} = \frac{2 \pm \sqrt{7}}{3}.$$

$$\text{Also } -1 < \frac{2 - \sqrt{7}}{3} < \frac{2 + \sqrt{7}}{3} < 2 \Rightarrow \frac{2 - \sqrt{7}}{3} \text{ and } \frac{2 + \sqrt{7}}{3} \text{ both lie in } (-1, 2).$$

So there exist two real numbers $\frac{2 - \sqrt{7}}{3}$ and $\frac{2 + \sqrt{7}}{3}$ in $(-1, 2)$ such that

$$f'\left(\frac{2 - \sqrt{7}}{3}\right) = 0 \text{ and } f'\left(\frac{2 + \sqrt{7}}{3}\right) = 0.$$

Hence, Rolle's theorem is verified.

Example 2. Using Rolle's theorem, find the point on the curve $y = 16 - x^2$, $x \in [-1, 1]$ where the tangent is parallel to x -axis.

Solution. Given $y = 16 - x^2$ i.e. $f(x) = 16 - x^2$... (1)

- (i) As $f(x)$ is a polynomial function, it is continuous in $[-1, 1]$,
- (ii) $f(x)$ being a polynomial function is derivable in $(-1, 1)$ and
- (iii) $f(-1) = 16 - (-1)^2 = 15$, $f(1) = 16 - 1^2 = 15 \Rightarrow f(-1) = f(1)$.

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number c in $(-1, 1)$ such that $f'(c) = 0$.

Differentiating (1) w.r.t. x , we get $f'(x) = -2x$.

Now $f'(c) = 0 \Rightarrow -2c = 0 \Rightarrow c = 0$.

So there exists $0 \in (-1, 1)$ where $f'(c) = 0$ i.e. the tangent is parallel to x -axis.

From (1), when $x = 0$, $y = 16 - 0^2 = 16$.

Hence, there exists the point $(0, 16)$ on the given curve where the tangent is parallel to x -axis.

Example 3. Verify Rolle's theorem for the following functions and find point (or points) in the given interval where derivative is zero :

$$(a) f(x) = \sin x + \cos x - 1 \text{ in } \left[0, \frac{\pi}{2}\right]$$

$$(b) f(x) = \sin x - \sin 2x \text{ in } [0, \pi]$$

$$(c) f(x) = e^{2x} (\sin 2x - \cos 2x) \text{ in } \left[\frac{\pi}{8}, \frac{5\pi}{8}\right] \quad (\text{I.S.C. 2006})$$

$$(d) f(x) = e^{1-x^2} \text{ in } [-1, 1].$$

Solution. (a) Given $f(x) = \sin x + \cos x - 1$... (1)

(i) $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$,

(ii) $f(x)$ is derivable in $\left(0, \frac{\pi}{2}\right)$ and

(iii) $f(0) = \sin 0 + \cos 0 - 1 = 0 + 1 - 1 = 0$,

$$f\left(\frac{\pi}{2}\right) = \sin \frac{\pi}{2} + \cos \frac{\pi}{2} - 1 = 1 + 0 - 1 = 0 \Rightarrow f(0) = f\left(\frac{\pi}{2}\right).$$

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number c in $\left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Differentiating (1) w.r.t. x , we get

$$f'(x) = \cos x - \sin x$$

Now $f'(c) = 0 \Rightarrow \cos c - \sin c = 0 \Rightarrow \tan c = 1$

$$\Rightarrow c = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, -\frac{3\pi}{4}, \dots \text{ but } c \in \left(0, \frac{\pi}{2}\right) \Rightarrow c = \frac{\pi}{4}.$$

So there exists $\frac{\pi}{4}$ in $\left(0, \frac{\pi}{2}\right)$ such that $f'\left(\frac{\pi}{4}\right) = 0$.

Hence, Rolle's theorem is verified and $c = \frac{\pi}{4}$.

(b) Given $f(x) = \sin x - \sin 2x$... (1)

(i) $f(x)$ is continuous in $[0, \pi]$.

(ii) $f(x)$ is derivable in $(0, \pi)$ and

(iii) $f(0) = \sin 0 - \sin 0 = 0 - 0 = 0$, $f(\pi) = \sin \pi - \sin 2\pi = 0 - 0 = 0$

$\Rightarrow f(0) = f(\pi)$.

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number c in $(0, \pi)$ such that $f'(c) = 0$.

Diff. (1) w.r.t. x , we get $f'(x) = \cos x - \cos 2x$.

Now $f'(c) = 0 \Rightarrow \cos c - 2 \cos 2c = 0$

$$\Rightarrow \cos c - 2(2 \cos^2 c - 1) = 0 \Rightarrow 4 \cos^2 c - \cos c - 2 = 0$$

$$\Rightarrow \cos c = \frac{1 \pm \sqrt{1 - 4 \cdot 4 \cdot (-2)}}{2 \cdot 4} = \frac{1 \pm \sqrt{33}}{8} \Rightarrow c = \cos^{-1} \left(\frac{1 \pm \sqrt{33}}{8} \right).$$

So there exist two real numbers ' c ' given by $c = \cos^{-1} \left(\frac{1 \pm \sqrt{33}}{8} \right)$ in $(0, \pi)$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified and $c = \cos^{-1} \left(\frac{1 \pm \sqrt{33}}{8} \right)$.

(c) Given $f(x) = e^{2x} (\sin 2x - \cos 2x)$... (1)

(i) $f(x)$ is continuous in $\left[\frac{\pi}{8}, \frac{5\pi}{8}\right]$,

(ii) $f(x)$ is derivable in $\left(\frac{\pi}{8}, \frac{5\pi}{8}\right)$ and

$$(iii) f\left(\frac{\pi}{8}\right) = e^{\frac{\pi}{4}} \left(\sin \frac{\pi}{4} - \cos \frac{\pi}{4}\right) = e^{\frac{\pi}{4}} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 0,$$

$$f\left(\frac{5\pi}{8}\right) = e^{\frac{5\pi}{4}} \left(\sin \frac{5\pi}{4} - \cos \frac{5\pi}{4}\right) = e^{\frac{5\pi}{4}} \left(-\sin \frac{\pi}{4} + \cos \frac{\pi}{4}\right) = e^{\frac{5\pi}{4}} \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = 0$$

$$\Rightarrow f\left(\frac{\pi}{8}\right) = f\left(\frac{5\pi}{8}\right).$$

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number c in $\left(\frac{\pi}{8}, \frac{5\pi}{8}\right)$ such that $f'(c) = 0$.

Diff. (1) w.r.t. x , we get

$$f'(x) = e^{2x} (\cos 2x \cdot 2 + \sin 2x \cdot 2) + (\sin 2x - \cos 2x) e^{2x} \cdot 2 \\ = 4e^{2x} \sin 2x.$$

$$\text{Now } f'(c) = 0 \Rightarrow 4e^{2c} \sin 2c = 0 \Rightarrow \sin 2c = 0$$

$$\Rightarrow 2c = 0, \pi, 2\pi, \dots, -\pi, -2\pi, \dots$$

$$\Rightarrow c = 0, \frac{\pi}{2}, \pi, \dots, -\frac{\pi}{2}, -\pi, \dots \text{ but } c \in \left(\frac{\pi}{8}, \frac{5\pi}{8}\right) \Rightarrow c = \frac{\pi}{2}.$$

So there exists $\frac{\pi}{2}$ in $\left(\frac{\pi}{8}, \frac{5\pi}{8}\right)$ such that $f'\left(\frac{\pi}{2}\right) = 0$.

Hence, Rolle's theorem is verified and $c = \frac{\pi}{2}$.

(d) Given $f(x) = e^{1-x^2}$... (1)

(i) $f(x)$ is continuous in $[-1, 1]$, for,

since $g(x) = 1 - x^2$ and $h(x) = e^x$ are continuous in $[-1, 1]$ therefore,

$$(hog)(x) = h(g(x)) = h(1 - x^2) = e^{1-x^2} \text{ is also continuous in } [-1, 1],$$

(ii) $f(x)$ is derivable in $(-1, 1)$ and

$$(iii) f(-1) = e^{1-1} = e^0 = 1, f(1) = e^{1-1} = e^0 = 1 \Rightarrow f(-1) = f(1).$$

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number c in $(-1, 1)$ such that $f'(c) = 0$.

Differentiating (1) w.r.t. x , we get

$$f'(x) = e^{1-x^2} (-2x) = -2x e^{1-x^2}.$$

$$\text{Now } f'(c) = 0 \Rightarrow -2c e^{1-c^2} = 0 \Rightarrow c = 0.$$

So there exists 0 in $(-1, 1)$ such that $f'(0) = 0$.

Hence, Rolle's theorem is verified and $c = 0$.

Example 4. Verify Rolle's theorem for the following functions and find point (or points) in the given interval where derivative is zero :

$$(a) f(x) = (x - a)^m (x - b)^n \text{ in } [a, b], m, n \in \mathbf{N}.$$

$$(b) f(x) = \log \left(\frac{x^2 + ab}{(a+b)x} \right) \text{ in } [a, b], a > 0. \quad (\text{I.S.C. 2012})$$

Solution. (a) Given $f(x) = (x - a)^m (x - b)^n$, $m, n \in \mathbf{N}$... (1)

Since $m, n \in \mathbf{N}$, $f(x)$ is a polynomial in x .

(i) $f(x)$ is continuous in $[a, b]$,

(ii) $f(x)$ is derivable in (a, b) and

$$(iii) f(a) = 0, f(b) = 0 \Rightarrow f(a) = f(b).$$

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number c in (a, b) such that $f'(c) = 0$.

Differentiating (1) w.r.t. x , we get

$$\begin{aligned} f'(x) &= (x-a)^m n (x-b)^{n-1} + (x-b)^n m (x-a)^{m-1} \\ &= (x-a)^{m-1} (x-b)^{n-1} (n(x-a) + m(x-b)) \\ &= (x-a)^{m-1} (x-b)^{n-1} ((m+n)x - (na+mb)). \end{aligned}$$

$$\text{Now } f'(c) = 0 \Rightarrow (c-a)^{m-1} (c-b)^{n-1} ((m+n)c - (na+mb)) = 0.$$

$$\text{But } c \neq a, c \neq b \Rightarrow (m+n)c - (na+mb) = 0$$

$$\Rightarrow c = \frac{mb+na}{m+n},$$

which is a point in (a, b) , for, it divides $[a, b]$ in the ratio $m : n$ internally.

Thus, there exists a real number $c = \frac{mb+na}{m+n}$ in (a, b) such that $f'(c) = 0$.

Hence, Rolle's theorem is verified and $c = \frac{mb+na}{m+n}$.

$$(b) \text{ Given } f(x) = \log \left(\frac{x^2+ab}{(a+b)x} \right) = \log(x^2+ab) - \log(a+b) - \log x \quad \dots(1)$$

(i) Since $a > 0$ and $\log x$ is continuous for all $x > 0$, therefore, $f(x)$ is continuous in $[a, b]$,

(ii) $f(x)$ is derivable in (a, b) and

$$(iii) f(a) = \log \left(\frac{a^2+ab}{(a+b)a} \right) = \log 1 = 0, f(b) = \log \left(\frac{b^2+ab}{(a+b)b} \right) = \log 1 = 0$$

$$\Rightarrow f(a) = f(b).$$

Thus, all the three conditions of Rolle's theorem are satisfied, therefore, there exists atleast one real number c in (a, b) such that $f'(c) = 0$.

Differentiating (1) w.r.t. x , we get

$$f'(x) = \frac{1}{x^2+ab} \cdot 2x - 0 - \frac{1}{x} = \frac{2x}{x^2+ab} - \frac{1}{x} = \frac{x^2-ab}{x(x^2+ab)}.$$

$$\text{Now } f'(c) = 0 \Rightarrow \frac{c^2-ab}{c(c^2+ab)} = 0 \Rightarrow c^2-ab=0 \Rightarrow c = \pm\sqrt{ab}.$$

$$\text{But } c \in (a, b) \Rightarrow c = \sqrt{ab} \quad (\because \text{Geometric mean lies between them})$$

So there exists a real number $c = \sqrt{ab}$ in (a, b) such that $f'(c) = 0$.

Hence, Rolle's theorem is verified and $c = \sqrt{ab}$.

Example 5. It is given that for the function $f(x) = x^3 + bx^2 + ax + 5$ on $[1, 3]$ Rolle's theorem holds with $c = 2 + \frac{1}{\sqrt{3}}$. Find the values of a and b .

$$\text{Solution. Given } f(x) = x^3 + bx^2 + ax + 5 \quad \dots(1)$$

We note that $f(x)$ is continuous in $[1, 3]$ and derivable in $(1, 3)$ for all values of a and b .

Differentiating (1) w.r.t. x , we get

$$f'(x) = 3x^2 + 2bx + a \quad \dots(2)$$

Since Rolle's theorem holds for $f(x)$ in $[1, 3]$ with $c = 2 + \frac{1}{\sqrt{3}}$, therefore, we must have

$$f(1) = f(3) \text{ and } f'\left(2 + \frac{1}{\sqrt{3}}\right) = 0$$

$$\Rightarrow 1 + b + a + 5 = 27 + 9b + 3a + 5 \Rightarrow 8b + 2a + 26 = 0$$

$$\Rightarrow a + 4b + 13 = 0 \quad \dots(3)$$

$$\text{and } 3\left(2 + \frac{1}{\sqrt{3}}\right)^2 + 2b\left(2 + \frac{1}{\sqrt{3}}\right) + a = 0$$

$$\Rightarrow 3\left(4 + \frac{1}{3} + \frac{4}{\sqrt{3}}\right) + 4b + \frac{2b}{\sqrt{3}} + a = 0$$

$$\Rightarrow (a + 4b + 13) + \frac{12}{\sqrt{3}} + \frac{2b}{\sqrt{3}} = 0 \Rightarrow \frac{12}{\sqrt{3}} + \frac{2b}{\sqrt{3}} = 0 \quad (\text{using (3)})$$

$$\Rightarrow 2b + 12 = 0 \Rightarrow b = -6.$$

$$\text{From (3), } a - 24 + 13 = 0 \Rightarrow a = 11.$$

$$\text{Hence, } a = 11, b = -6.$$

Example 6. Discuss the applicability of Rolle's theorem for the function $f(x) = |x|$ in $[-2, 2]$.

Solution. Given $f(x) = |x|$, $x \in [-2, 2]$... (1)

the graph of $f(x) = |x|$ in $[-2, 2]$

is shown in fig. 8.2.

(i) $f(x)$ is continuous in $[-2, 2]$.

(ii) Differentiating (1) w.r.t. x , we get

$$f'(x) = \frac{x}{|x|}, x \neq 0$$

$\Rightarrow$ the derivative of $f(x)$ does not exist at $x = 0$

$\Rightarrow f(x)$ is not derivable in $(-2, 2)$.

Thus, the condition (ii) of Rolle's theorem is not satisfied, therefore, Rolle's theorem is not applicable to the function $f(x) = |x|$ in $[-2, 2]$.

Moreover, $f(-2) = |-2| = 2$ and $f(2) = |2| = 2 \Rightarrow f(-2) = f(2)$, so the condition (iii) of Rolle's theorem is satisfied.

Further, it is clear from the graph that there is no point of the curve $y = |x|$ in $(-2, 2)$ at which the tangent is parallel to x -axis.

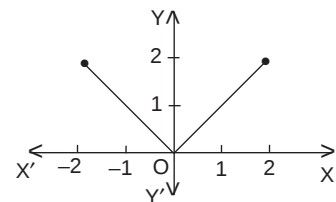


Fig. 8.2.

EXERCISE 8.1

Verify Rolle's theorem for the following (1 to 6) functions and find point (or points) in the interval where derivative is zero:

1. (i) $f(x) = x^2 - 5x + 4$ in $[1, 4]$ (ii) $f(x) = x^2 + 5x + 6$ in $[-3, -2]$
(iii) $f(x) = x^2 - 8x + 12$ in $[2, 6]$.
2. (i) $f(x) = (x - 1)(x - 2)^2$ in $[1, 2]$ (ii) $f(x) = (x - 1)(x - 2)(x - 3)$ in $[1, 3]$
(iii) $f(x) = x^3 - 12x$ in $[0, 2\sqrt{3}]$ (iv) $f(x) = x^3 - 4x$ in $[-2, 2]$
(v) $f(x) = 2x^3 + x^2 - 4x - 2$ in $\left[-\frac{1}{2}, \sqrt{2}\right]$.
3. (i) $f(x) = \cos 2x$ in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ (ii) $f(x) = \sin x - 1$ in $\left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$.
4. (i) $f(x) = \sin 2x$ in $\left[0, \frac{\pi}{2}\right]$ (ii) $f(x) = \sin 3x$ in $[0, \pi]$
(iii) $f(x) = \sin^2 x$ in $[0, \pi]$ (iv) $f(x) = \cos 2\left(x - \frac{\pi}{4}\right)$ in $\left[0, \frac{\pi}{2}\right]$.
5. (i) $f(x) = \log(x^2 + 2) - \log 3$ on $[-1, 1]$
(ii) $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ (iii) $f(x) = \sin x + \cos x$ on $\left[0, \frac{\pi}{2}\right]$
(iv) $f(x) = |9 - x^2|$ on $[-3, 3]$.
6. (i) $f(x) = e^x \sin x$ on $[0, \pi]$ (I.S.C. 2005)
(ii) $f(x) = e^x \cos x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (iii) $f(x) = \frac{\sin x}{e^x}$ on $[0, \pi]$
(iv) $f(x) = e^x (\sin x - \cos x)$ on $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

7. Apply Rolle's theorem to find point (or points) on the following curves where the tangent is parallel to x -axis:

- (i) $y = x^2$ in $[-2, 2]$ (ii) $f(x) = -1 + \cos x$ in $[0, 2\pi]$.

8. Using Rolle's theorem, find a point on the curve $y = \sin x + \cos x - 1$, $x \in \left[0, \frac{\pi}{2}\right]$ where the tangent is parallel to the x -axis. (I.S.C. 2010)
9. If Rolle's theorem holds for the function $f(x) = x^3 + ax^2 + bx$ in $[1, 2]$ at the point $x = \frac{4}{3}$, then find the values of a and b . (I.S.C. 2009)
10. Considering the function $f(x) = (x - 3) \log x$, prove that there is atleast one real value of x in $(1, 3)$ which satisfies $x \log x = 3 - x$.
11. Discuss the applicability of Rolle's theorem for the following functions in the indicated intervals :
- (i) $f(x) = x^{1/3}$ in $[-1, 1]$ (ii) $f(x) = x^{2/3}$ in $[-1, 1]$.
12. Discuss the applicability of Rolle's theorem for the following functions in the indicated intervals :
- (i) $f(x) = \tan x$ in $[0, \pi]$ (ii) $f(x) = \sec x$ in $[0, 2\pi]$
- (iii) $f(x) = 2 + (x - 1)^{2/3}$ in $[0, 2]$ (iv) $f(x) = 1 + |x - 2|$ in $[0, 4]$.

8.2 LAGRANGE'S MEAN VALUE THEOREM

If a function $f(x)$ is

(i) continuous in the closed interval $[a, b]$ and

(ii) derivable in the open interval (a, b) ,

then there exists atleast one real number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Consider a function g defined by $g(x) = f(x) + kx$... (1)

where k is a constant (real number) to be chosen in such a way that $g(a) = g(b)$

$$\Rightarrow f(c) + ka = f(b) + kb \Rightarrow k(a - b) = f(b) - f(a)$$

$$\Rightarrow k = -\frac{f(b) - f(a)}{b - a} \quad \dots (2)$$

Now (i) $g(x)$ is continuous in $[a, b]$

($\because f(x)$ is given to be continuous in $[a, b]$ and kx being polynomial function is continuous for all $x \in \mathbf{R} \Rightarrow$ their sum i.e. $f(x) + kx$ is continuous in $[a, b]$)

(ii) $g(x)$ is derivable in (a, b)

($\because f(x)$ is given to be derivable in (a, b) and kx being polynomial function is derivable for all $x \in \mathbf{R} \Rightarrow$ their sum i.e. $f(x) + kx$ is derivable in (a, b))

(iii) $g(a) = g(b)$.

Thus, all the three conditions of Rolle's theorem are satisfied by the function g in $[a, b]$, therefore, there exists atleast one real number c in (a, b) such that $g'(c) = 0$.

Differentiating (1) w.r.t. x , we get

$$g'(x) = f'(x) + k.1$$

$$\text{Now } g'(c) = 0 \Rightarrow f'(c) + k = 0 \Rightarrow k = -f'(c) \quad \dots (3)$$

From (2) and (3), we get

$$-f'(c) = -\frac{f(b) - f(a)}{b - a} \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Geometrical Interpretation

Let A, B be the points on the curve $y = f(x)$ corresponding to the real numbers a, b respectively.

Since $f(x)$ is continuous in $[a, b]$, the graph of the curve $y = f(x)$ is continuous from A to B. Again, as $f(x)$ is derivable in (a, b) the curve $y = f(x)$ has a tangent at each point between A and B. Also as $a \neq b$, the slope of the chord AB exists and the slope of the chord

$$AB = \frac{f(b) - f(a)}{b - a}.$$

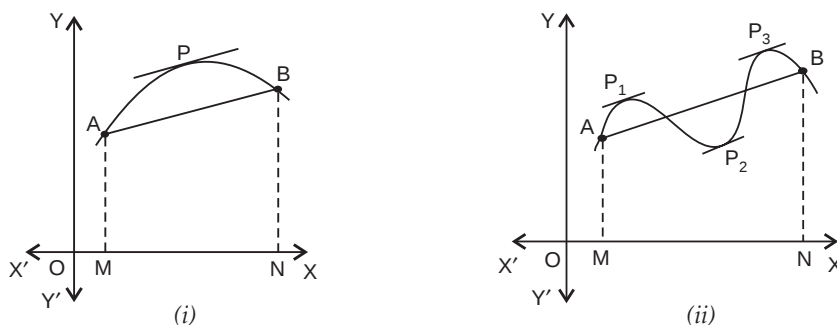


Fig. 8.3.

Then *Lagrange's Mean Value Theorem* asserts that there is atleast one point lying between A and B such that the tangent at which is parallel to the chord AB. There may exist more than one point between A and B the tangents at which are parallel to the chord AB (as shown in fig. 8.3 (ii)). Lagrange's mean value theorem ensures the existence of atleast one real number c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Remarks

1. Lagrange's mean value theorem fails for the function which does not satisfy even one of the two conditions.
2. The converse of Lagrange's mean value theorem may not be true, for, $f'(c)$ may be equal to $\frac{f(b) - f(a)}{b - a}$ at a point c in (a, b) without satisfying both the conditions of Lagrange's mean value theorem (see example 4 on page 339).

ILLUSTRATIVE EXAMPLES

Example 1. Verify Lagrange's mean value theorem for the following functions in the given interval and find 'c' of this theorem.

(a) $f(x) = 3x^2 - 5x + 1$ in $[2, 5]$

(I.S.C. 2007)

(b) $f(x) = (x - 1)(x - 2)(x - 3)$ in $[0, 4]$.

Solution. (a) Given $f(x) = 3x^2 - 5x + 1$, $x \in [2, 5]$

...(1)

(i) $f(x)$ being a polynomial function is continuous in $[2, 5]$

(ii) $f(x)$ being a polynomial function is derivable in $(2, 5)$.

Thus, both the conditions of Lagrange's mean value theorem are satisfied, therefore, there exists atleast one real number c in $(2, 5)$ such that

$$f'(c) = \frac{f(5) - f(2)}{5 - 2}.$$

$$f(5) = 3 \cdot 5^2 - 5 \cdot 5 + 1 = 51, f(2) = 3 \cdot 2^2 - 5 \cdot 2 + 1 = 3.$$

Differentiating (1) w.r.t. x , we get

$$f'(x) = 3 \cdot 2x - 5 \cdot 1 + 0 \Rightarrow f'(c) = 6c - 5.$$

$$\therefore f'(c) = \frac{f(5) - f(2)}{5 - 2} \Rightarrow 6c - 5 = \frac{51 - 3}{3} \Rightarrow 6c - 5 = 16$$

$$\Rightarrow 6c = 21 \Rightarrow c = \frac{7}{2}.$$

$$\text{Thus, there exists } c = \frac{7}{2} \text{ in } (2, 5) \text{ such that } f'\left(\frac{7}{2}\right) = \frac{f(5) - f(2)}{5 - 2}.$$

Hence, Lagrange's mean value theorem is verified and $c = \frac{7}{2}$.

(b) Given $f(x) = (x - 1)(x - 2)(x - 3) = x^3 - 6x^2 + 11x - 6$

...(1)

(i) $f(x)$ being a polynomial function is continuous in $[0, 4]$

(ii) $f(x)$ being a polynomial function is derivable in $(0, 4)$.

Thus, both the conditions of Lagrange's mean value theorem are satisfied, therefore, there exists atleast one real number c in $(0, 4)$ such that

$$f'(c) = \frac{f(4) - f(0)}{4 - 0}.$$

$$f(4) = 64 - 96 + 44 - 6 = 6, f(0) = 0 - 0 + 0 - 6 = -6.$$

Differentiating (1) w.r.t. x , we get

$$f'(x) = 3x^2 - 12x + 11 \Rightarrow f'(c) = 3c^2 - 12c + 11$$

$$\therefore f'(c) = \frac{f(4) - f(0)}{4 - 0} \Rightarrow 3c^2 - 12c + 11 = \frac{6 - (-6)}{4 - 0}$$

$$\Rightarrow 3c^2 - 12c + 11 = 3 \Rightarrow 3c^2 - 12c + 8 = 0$$

$$\Rightarrow c = \frac{12 \pm \sqrt{144 - 4 \cdot 3 \cdot 8}}{2 \cdot 3} = 2 \pm \frac{\sqrt{48}}{6} = 2 \pm \frac{2}{\sqrt{3}}.$$

$$\text{As } 0 < 2 - \frac{2}{\sqrt{3}} < 2 + \frac{2}{\sqrt{3}} < 4 \Rightarrow 2 - \frac{2}{\sqrt{3}} \text{ and } 2 + \frac{2}{\sqrt{3}} \text{ both lie in } (0, 4).$$

$$\text{Thus, there exist } c = 2 \pm \frac{2}{\sqrt{3}} \text{ in } (0, 4) \text{ such that } f'(c) = \frac{f(4) - f(0)}{4 - 0}.$$

Hence, Lagrange's mean value theorem is verified and $c = 2 \pm \frac{2}{\sqrt{3}}$.

Example 2. Verify Lagrange's mean value theorem for the following functions in the given intervals :

$$(a) f(x) = \tan^{-1} x \text{ in } [0, 1] \quad (b) f(x) = \sqrt{x^2 - x} \text{ in } [1, 4]. \quad (\text{I.S.C. 2013})$$

Solution. (a) Given $f(x) = \tan^{-1} x$, $x \in [0, 1]$... (1)

(i) $f(x)$ is continuous in $[0, 1]$

(ii) $f(x)$ is derivable in $(0, 1)$.

Thus, both the conditions of Lagrange's mean value theorem are satisfied, therefore, there exists atleast one real number c in $(0, 1)$ such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$$f(1) = \tan^{-1}(1) = \frac{\pi}{4}, f(0) = \tan^{-1}(0) = 0.$$

Differentiating (1) w.r.t. x , we get

$$f'(x) = \frac{1}{1+x^2} \Rightarrow f'(c) = \frac{1}{1+c^2}$$

$$\therefore f'(c) = \frac{f(1) - f(0)}{1 - 0} \Rightarrow \frac{1}{1+c^2} = \frac{\frac{\pi}{4} - 0}{1} \Rightarrow \frac{1}{1+c^2} = \frac{\pi}{4}$$

$$\Rightarrow 1 + c^2 = \frac{4}{\pi} \Rightarrow c^2 = \frac{4}{\pi} - 1 \Rightarrow c = \pm \sqrt{\frac{4 - \pi}{\pi}}.$$

$$\text{But } c \in (0, 1) \Rightarrow c = \sqrt{\frac{4 - \pi}{\pi}}.$$

$$\text{Thus, there exists } c = \sqrt{\frac{4 - \pi}{\pi}} \text{ in } (0, 1) \text{ such that } f'(c) = \frac{f(1) - f(0)}{1 - 0}.$$

Hence, Lagrange's mean value theorem is verified.

(b) Given $f(x) = \sqrt{x^2 - x}$, $x \in [1, 4]$... (1)

(i) Since $g(x) = x^2 - x$ is continuous on $\mathbf{R}$ and $h(x) = \sqrt{x}$ is continuous in $[0, \infty)$, therefore,

(hog) $(x) = h(g(x)) = h(x^2 - x) = \sqrt{x^2 - x}$ is continuous for all x such that $x^2 - x \geq 0$

$$\Rightarrow \sqrt{x^2 - x} \text{ is continuous in } (-\infty, 0] \cup [1, \infty)$$

$$\Rightarrow \sqrt{x^2 - x} \text{ is continuous in } [1, 4].$$

(ii) Differentiating (1) w.r.t. x , we get

$$f'(x) = \frac{1}{2}(x^2 - x)^{-1/2} (2x - 1) = \frac{2x - 1}{2\sqrt{x^2 - x}}$$

which exists for all x such that $x^2 - x > 0$ i.e. for x in $(-\infty, 0) \cup (1, \infty)$

$\Rightarrow f(x)$ is derivable for all x in $(1, 4)$.

Thus, both the conditions of Lagrange's mean value theorem are satisfied, therefore, there exists atleast one real number c in $(1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}$$

$$\Rightarrow \frac{2c - 1}{2\sqrt{c^2 - c}} = \frac{\sqrt{4^2 - 4} - \sqrt{1^2 - 1}}{4 - 1} = \frac{\sqrt{12}}{3} = \frac{2}{\sqrt{3}}$$

$$\Rightarrow 3(2c - 1)^2 = 16(c^2 - c) \Rightarrow 4c^2 - 4c - 3 = 0$$

$$\Rightarrow (2c - 3)(2c + 1) = 0 \Rightarrow c = \frac{3}{2}, -\frac{1}{2}.$$

Thus, there exists $c = \frac{3}{2} \in (1, 4)$ such that $f'(c) = \frac{f(4) - f(1)}{4 - 1}$.

Hence, Lagrange's mean value theorem is verified.

Example 3. Find a point on the graph of $y = x^3$ where the tangent is parallel to the chord joining $(1, 1)$ and $(3, 27)$.

Solution. Let us apply Lagrange's mean value theorem to the function

$$f(x) = x^3 \text{ in the interval } [1, 3] \quad \dots(1)$$

(i) $f(x)$ being polynomial is continuous in $[1, 3]$

(ii) $f(x)$ being polynomial is derivable in $(1, 3)$.

Thus, both the conditions of Lagrange's mean value theorem are satisfied by the function $f(x)$ in $[1, 3]$, therefore, there exists atleast one real number c in $(1, 3)$ such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}.$$

$$f(3) = 3^3 = 27 \text{ and } f(1) = 1^3 = 1.$$

Differentiating (1) w.r.t. x , we get

$$f'(x) = 3x^2 \Rightarrow f'(c) = 3c^2.$$

$$\text{Now } f'(c) = \frac{f(3) - f(1)}{3 - 1} \Rightarrow 3c^2 = \frac{27 - 1}{3 - 1} \Rightarrow 3c^2 = 13$$

$$\Rightarrow c^2 = \frac{13}{3} = \frac{39}{9} \Rightarrow c = \pm \frac{\sqrt{39}}{3}.$$

$$\text{But } c \in (1, 3) \Rightarrow c = \frac{\sqrt{39}}{3}.$$

$$\text{When } x = \frac{\sqrt{39}}{3}, \text{ from (1), } y = \frac{13\sqrt{39}}{9}.$$

Hence, there exists a point $\left(\frac{\sqrt{39}}{3}, \frac{13\sqrt{39}}{9}\right)$ on the given curve $y = x^3$ where the tangent is parallel to the chord joining the points $(1, 1)$ and $(3, 27)$.

Example 4. Does the Lagrange's mean value theorem apply to $f(x) = x^{1/3}$, $-1 \leq x \leq 1$? What conclusions can be drawn?

$$\text{Solution. Given } f(x) = x^{1/3}, x \in [-1, 1] \quad \dots(1)$$

(i) $f(x)$ is continuous in $[-1, 1]$.

(ii) Differentiating (1) w.r.t. x , we get

$$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}, x \neq 0 \quad \dots(2)$$

9. Using Lagrange's mean value theorem, find a point on the curve $y = \sqrt{x-2}$ defined in the interval $[2, 3]$ where the tangent is parallel to the chord joining the end points of the curve. (I.S.C. 2008)
10. Discuss the applicability of Lagrange's mean value theorem for the following functions in the indicated intervals.
- (i) $f(x) = |x|$ in $[-1, 1]$ (ii) $f(x) = 1 - (2-x)^{2/3}$ in $[1, 3]$.

8.3 MAXIMA AND MINIMA

8.3.1 Absolute maxima and absolute minima

Let f be a real valued function defined on D (subset of $\mathbf{R}$), then

(i) f is said to have **absolute maxima** at $x = c$ (in D) iff $f(x) \leq f(c)$ for all $x \in D$, and c is called point of **absolute maxima** and $f(c)$ is called **absolute maximum** (or **greatest**) value of f on D .

(ii) f is said to have **absolute minima** at $x = d$ (in D) iff $f(d) \leq f(x)$ for all $x \in D$, and d is called point of **absolute minima** and $f(d)$ is called **absolute minimum** (or **smallest**) value of f on D .

Obviously, **absolute maximum** and **absolute minimum** values of a function (if they exist) are unique. However, absolute maximum or absolute minimum values of a given function f on D (subset of $\mathbf{R}$) may be obtained at more than one points. Also it is not essential that a given function must have **maxima** or **minima** in its domain.

For example :

(i) Consider the function $f(x) = \sin x$ in $\left[-\pi, \frac{3\pi}{2}\right]$.

Its graph is shown in fig. 8.5. Here $c = \frac{\pi}{2}$ is a point of maxima and maximum value $= f\left(\frac{\pi}{2}\right) = \sin \frac{\pi}{2} = 1$.

There are two points of minima,

$d = -\frac{\pi}{2}$ or $\frac{3\pi}{2}$ and the minimum value

$$= f\left(-\frac{\pi}{2}\right) = \sin\left(-\frac{\pi}{2}\right) = -\sin \frac{\pi}{2} = -1$$

$$\text{or } f\left(\frac{3\pi}{2}\right) = \sin \frac{3\pi}{2} = -\sin \frac{\pi}{2} = -1.$$

(ii) Consider the function

$$f(x) = \cos x \text{ in } \left[-\frac{\pi}{2}, 2\pi\right].$$

Its graph is shown in fig. 8.6.

Here $d = \pi$ is a point of minima and minimum value $= f(\pi) = \cos \pi = -1$.

There are two points of maxima, $c = 0, 2\pi$ and the maximum value

$$= f(0) = \cos 0 = 1 \text{ or } f(2\pi) = \cos 2\pi = \cos 0 = 1.$$

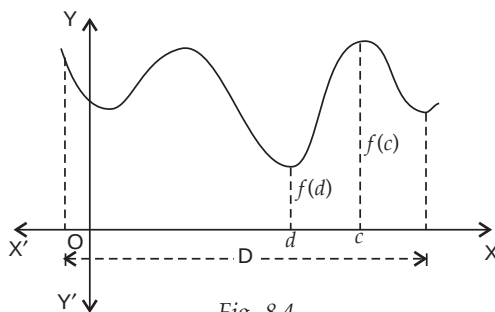


Fig. 8.4.

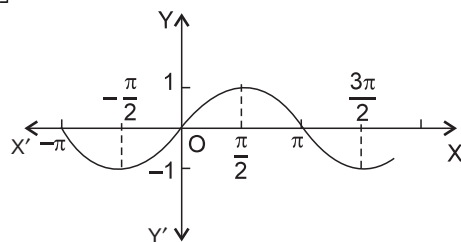


Fig. 8.5.

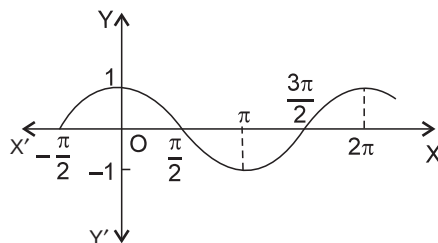


Fig. 8.6.

(iii) Consider the function $f(x) = x^2$,

$$D_f = \mathbf{R} \text{ and } R_f = [0, \infty).$$

The graph of f is shown in fig. 8.7.

The function f has minima at $x = 0$ and minimum value $= f(0) = 0$. Note that f has no maxima.

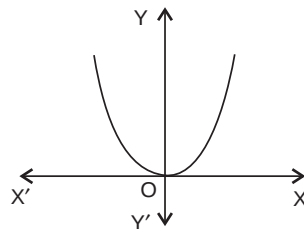


Fig. 8.7.

(iv) Consider the function $f(x) = 2x - x^2$,

$$D_f = \mathbf{R}. \text{ It can be written as}$$

$$y = 2x - x^2 = 1 - (x - 1)^2,$$

$$R_f = (-\infty, 1].$$

The graph of f is shown in fig. 8.8.

The function f has maxima at $x = 1$ and maximum value $= f(1) = 2 \cdot 1 - 1^2 = 1$.

Note that f has no minima.

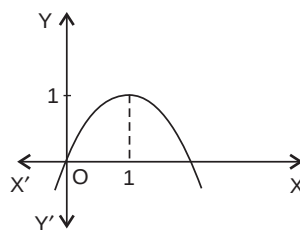


Fig. 8.8.

8.3.2 Local maxima and local minima

Let f be a real valued function defined on D (subset of $\mathbf{R}$), then

(i) f is said to have a **local** (or **relative**) **maxima** at $x = c$ (in D) iff there exists a positive real number δ such that $f(x) \leq f(c)$ for all x in $(c - \delta, c + \delta)$ i.e. $f(x) \leq f(c)$ for all x in the immediate neighbourhood of c , and c is called point of **local maxima** and $f(c)$ is called **local maximum value**.

(ii) f is said to have **local** (or **relative**) **minima** at $x = d$ (in D) iff there exists some positive real number δ such that $f(d) \leq f(x)$ for all $x \in (d - \delta, d + \delta)$ i.e. $f(d) \leq f(x)$ for all x in the immediate neighbourhood of d , and d is called point of **local minima** and $f(d)$ is called **local minimum value**.

Geometrically, a point c in the domain of the given function f is a point of **local maxima** or **local minima** according as the graph of f has a **peak** or **trough (cavity)** at c .

(iii) a point (in D) which is either a point of local maxima or a point of local minima is called an **extreme point**, and the value of the function at this point is called an **extreme value**.

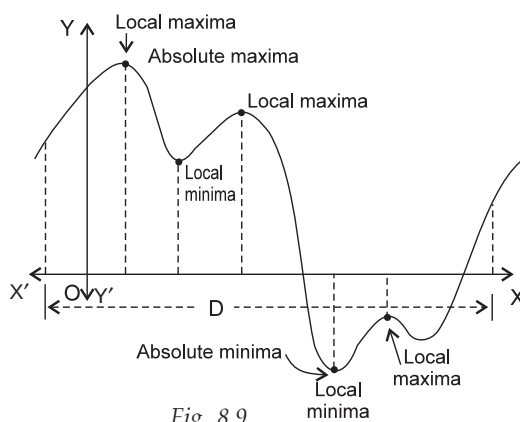


Fig. 8.9.

Remarks

1. A local maximum (minimum) value may not be the absolute maximum (minimum) value.
2. A local maximum value at some point may be less than a local minimum value of the function at another point.

Stationary (or Turning) Point

Let f be a real valued function defined on D (subset of $\mathbf{R}$), then a point c (in D) is called a **stationary** (or **turning** or **critical**) **point** of f iff f is differentiable at $x = c$ and $f'(c) = 0$.

However, it is not essential that an extreme point is a stationary point, and a stationary point is an extreme point.

For example :

(i) Consider the function $f(x) = \sin x$, $D_f = \mathbf{R}$. A portion of the graph of this function is shown in fig. 8.5.

11. Find the area of the largest isosceles triangle having perimeter 18 metres.

Hint. Let x be one of two equal sides, then base = $18 - 2x$, then

area = $A = \sqrt{9(9-x)(9-x)(9-(18-2x))}$, $\frac{9}{2} < x < 9$, for, if $x < \frac{9}{2}$, then the sum of two sides is smaller than the third side.

12. Find a point on the hypotenuse of a given right-angled triangle from which the perpendiculars can be dropped on the other sides to form a rectangle of maximum area.

Hint. Let l be the length of the hypotenuse of the given right-angled triangle ABC at

B and $\angle CAB = \alpha$ (in radian measure), $0 < \alpha < \frac{\pi}{2}$.

As $\triangle ABC$ is given, l and α are fixed. Let P be a point on AC and $AP = x$, then $PC = l - x$.

13. Divide the number 4 into two positive numbers such that the sum of square of one and the cube of other is minimum.
14. The perimeter of a sector of a circle is p . Show that its area is maximum when its radius is $\frac{p}{4}$.
15. Find the minimum distance from the point $(4, 2)$ to the parabola $y^2 = 8x$.
16. Find the point on the curve $y^2 = 2x$ which is nearest to the point $(1, -4)$.
17. Find the dimensions of the rectangle of maximum area that can be inscribed in the portion of the parabola $y^2 = 4px$ intercepted by the line $x = a$.
18. A point on the hypotenuse of a right angled triangle is at distances a and b from the sides. Show that the minimum length of the hypotenuse is $(a^{2/3} + b^{2/3})^{3/2}$.
19. Show that the maximum volume of a cylinder which can be inscribed in a cone of height h and semi-vertical angle 30° is $\frac{4}{81}\pi h^3$.
20. A cylinder is such that the sum of its height and circumference of its base is 10 metres. Find the maximum volume of the cylinder.

Hint. If h metres be the height and r metres be the radius of the base of the cylinder, then $2\pi r + h = 10 \Rightarrow h = 10 - 2\pi r$.

V (volume of cylinder) = $\pi r^2 h = \pi r^2 (10 - 2\pi r)$.

21. Find the semi-vertical angle of the cone of maximum curved surface that can be inscribed in a sphere of radius R .

ANSWERS

EXERCISE 8.1

1. (i) $\frac{5}{2}$ (ii) $-\frac{5}{2}$ (iii) 4.
2. (i) $\frac{4}{3}$ (ii) $2 \pm \frac{1}{\sqrt{3}}$ (iii) 2 (iv) $\pm \frac{2}{\sqrt{3}}$ (v) $\frac{2}{3}$.
3. (i) 0 (ii) $\frac{3\pi}{2}$.
4. (i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ (iii) $\frac{\pi}{2}$ (iv) $\frac{\pi}{4}$.
5. (i) 0 (ii) 0 (iii) $\frac{\pi}{4}$ (iv) 0.
6. (i) $\frac{3\pi}{4}$ (ii) $\frac{\pi}{4}$ (iii) $\frac{\pi}{4}$ (iv) π .
7. (i) $(0, 0)$ (ii) $(\pi, -2)$.
8. $\left(\frac{\pi}{4}, \sqrt{2} - 1\right)$.
9. $a = -5, b = 8$.
11. Not applicable in both parts (i) and (ii), for :
 (i) not derivable at $x = 0$ (ii) not derivable at $x = 0$.

12. Not applicable in parts (i) to (iv), for:

- (i) discontinuous at $\frac{\pi}{2}$ (ii) discontinuous at $\frac{\pi}{2}, \frac{3\pi}{2}$
 (iii) not derivable at $x = 1$ (iv) not derivable at $x = 2$.

EXERCISE 8.2

1. (i) 2 (ii) 2 (iii) $\frac{9}{2}$ (iv) $\frac{-8 + 4\sqrt{13}}{3}$.
 2. (i) ± 1 (ii) $1 - \frac{\sqrt{21}}{6}$ (iii) $6 - \frac{\sqrt{39}}{3}$.
 3. (i) $\sqrt{3}$ (ii) $\frac{1 + 3\sqrt{5}}{4}$ (iii) $\frac{1}{3}(2 + \sqrt{13})$ (iv) $\frac{8}{27}$.
 4. (i) $\cos^{-1}\left(\frac{2}{\pi}\right)$ (ii) $\pm \frac{\pi}{2}$ (iii) $\frac{\pi}{3}$.
 5. (i) $\log_2 e$ (ii) $\sqrt{6}$ (iii) $\cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right)$.
 6. $\left(\frac{7}{2}, \frac{1}{4}\right)$. 7. $\left(\sqrt{\frac{7}{3}}, -\frac{2}{3}\sqrt{\frac{7}{3}}\right)$. 8. $(2, -7)$. 9. $\left(\frac{9}{4}, \frac{1}{2}\right)$.

10. Not applicable in both (i) and (ii), for:

- (i) not derivable at $x = 0$ (ii) not derivable at $x = 2$.

EXERCISE 8.3

1. (i) Minimum value = 3, no maximum value.
 (ii) Maximum value = 7, no minimum value.
 2. (i) Minimum value = -2, no maximum value.
 (ii) Maximum value = 9, no minimum value.
 3. (i) Neither maximum nor minimum.
 (ii) Minimum value = 0, no maximum value.
 4. (i) Maximum value = 3, no minimum value.
 (ii) Maximum value = 6, minimum value = 4.
 5. (i) Maximum value = 4, minimum value = 2.
 (ii) Maximum value = -2, minimum value = -3.
 6. (i) Maximum value = 5, minimum value = -1.
 (ii) Maximum value = 5, minimum value = -5.

7. -2.

8. 1.

9. $x = \frac{\pi}{4}$.

11. Maximum value	Minimum value	Point of maxima	Point of minima
(i) 8	-8	$x = 2$	$x = -2$
(ii) 19	3	$x = -3$	$x = 1$
(iii) 738	8	$x = 10$	$x = 0$
(iv) 25	-39	$x = 0$	$x = 2$
(v) 2π	0	$x = 2\pi$	$x = 0$
(vi) $\sqrt{2}$	-1	$x = \frac{\pi}{4}$	$x = \pi$
(vii) 7	3	$x = 3$	$x = -1$
(viii) $\frac{3}{2}$	-3	$x = \frac{\pi}{6}, \frac{5\pi}{6}$	$x = \frac{\pi}{2}$.

12. 18; -9.

13. $\frac{\pi}{4}, \frac{5\pi}{4}$.

14. Maximum value = 2π , minimum value = 0.