10 Definite Integrals

10.1 FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

Let f be a continuous function on the closed interval* [*a, b*] *and* ϕ *be an anti-derivative of f, then

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a). \qquad \text{(We assume it without proof)}$$

In words, the above theorem tells us that

$$\int_a^b f(x)\,dx = \text{(value of an anti-derivative at } b\text{, the upper limit)}$$
$$- \text{(value of the same anti-derivative at } a\text{, the lower limit).}$$

Remarks

1. We often write $\phi(b) - \phi(a)$ as $[\phi(x)]_a^b$.
2. No matter which anti-derivative we take as ϕ, the value of the definite integral comes out to be the same.

10.1.1 Evaluation of Definite Integrals

The fundamental theorem enables us to evaluate the definite integrals by making use of *anti-derivatives*.

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate the following :*

(*i*) $\int_0^1 (2x^3 + 3)^2\,dx$ (*ii*) $\int_3^5 \frac{dt}{1+3t}$.

Solution. (*i*) $\int_0^1 (2x^3 + 3)^2\,dx = \int_0^1 (4x^6 + 12x^3 + 9)\,dx$

$$= \left[4.\frac{x^7}{7} + 12.\frac{x^4}{4} + 9x\right]_0^1 = \left[\frac{4}{7}x^7 + 3x^4 + 9x\right]_0^1$$

$$= \left(\frac{4}{7} + 3 + 9\right) - (0 + 0 + 0) = \frac{88}{7}.$$

(*ii*) $$\int_3^5 \frac{dt}{1+3t} = \left[\frac{\log|1+3t|}{3}\right]_3^5 = \frac{1}{3}(\log 16 - \log 10)$$

$$= \frac{1}{3}\log\frac{16}{10} = \frac{1}{3}\log\frac{8}{5}.$$

Example 2. *Evaluate the following :*

(i) $\int_0^{\pi/2} \cos^2 x\, dx$ *(ii)* $\int_0^{\pi/2} \cos^4 x\, dx.$

Solution. (*i*) $\int_0^{\pi/2} \cos^2 x\, dx = \int_0^{\pi/2} \frac{1+\cos 2x}{2}\, dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2}$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{1}{2}\sin \pi\right) - \left(0 + \frac{1}{2}\sin 0\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{1}{2}.0\right) - \frac{1}{2}.0\right] = \frac{\pi}{4}.$$

(*ii*) $\int_0^{\pi/2} \cos^4 x\, dx = \int_0^{\pi/2} (\cos^2 x)^2\, dx = \int_0^{\pi/2} \left(\frac{1+\cos 2x}{2}\right)^2 dx$

$$= \frac{1}{4}\int_0^{\pi/2} (1 + 2\cos 2x + \cos^2 2x)\, dx = \frac{1}{4}\int_0^{\pi/2} \left(1 + 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx$$

$$= \frac{1}{8}\int_0^{\pi/2} (3 + 4\cos 2x + \cos 4x)\, dx = \frac{1}{8}\left[3x + 4.\frac{\sin 2x}{2} + \frac{\sin 4x}{4}\right]_0^{\pi/2}$$

$$= \frac{1}{8}\left[3\left(\frac{\pi}{2} - 0\right) + 2(\sin \pi - \sin 0) + \frac{1}{4}(\sin 2\pi - \sin 0)\right]$$

$$= \frac{1}{8}\left[\frac{3\pi}{2} + 2(0-0) + \frac{1}{4}(0-0)\right] = \frac{3\pi}{16}.$$

Example 3. *Evaluate the following :*

(i) $\int_0^{\pi/2} \sqrt{1+\cos 2x}\, dx$ *(ii)* $\int_0^{\pi/2} \sqrt{1+\sin 2x}\, dx.$

Solution. (*i*) $\int_0^{\pi/2} \sqrt{1+\cos 2x}\, dx = \int_0^{\pi/2} \sqrt{2\cos^2 x}\, dx = \sqrt{2}\int_0^{\pi/2} |\cos x|\, dx$

$$= \sqrt{2}\int_0^{\pi/2} \cos x\, dx$$

(As $0 \le x \le \frac{\pi}{2} \Rightarrow \cos x \ge 0 \Rightarrow |\cos x| = \cos x$)

$$= \sqrt{2}\left[\sin x\right]_0^{\pi/2} = \sqrt{2}\left[\sin \frac{\pi}{2} - \sin 0\right]$$

$$= \sqrt{2}(1-0) = \sqrt{2}.$$

(*ii*) $\int_0^{\pi/2} \sqrt{1+\sin 2x}\, dx = \int_0^{\pi/2} \sqrt{1+\cos\left(\frac{\pi}{2} - 2x\right)}\, dx = \int_0^{\pi/2} \sqrt{2\cos^2\left(\frac{\pi}{4} - x\right)}\, dx$

$$= \sqrt{2}\int_0^{\pi/2} \left|\cos\left(\frac{\pi}{4} - x\right)\right| dx = \sqrt{2}\int_0^{\pi/2} \cos\left(\frac{\pi}{4} - x\right) dx$$

$\Big[$As $0 \le x \le \frac{\pi}{2} \Rightarrow 0 \ge -x \ge -\frac{\pi}{2} \Rightarrow \frac{\pi}{4} \ge \frac{\pi}{4} - x \ge -\frac{\pi}{4}$

$\Rightarrow -\frac{\pi}{4} \le \frac{\pi}{4} - x \le \frac{\pi}{4} \Rightarrow \cos\left(\frac{\pi}{4} - x\right) > 0 \Rightarrow \left|\cos\left(\frac{\pi}{4} - x\right)\right| = \cos\left(\frac{\pi}{4} - x\right)\Big]$

$$= \sqrt{2}\left[\frac{\sin\left(\frac{\pi}{4}-x\right)}{-1}\right]_0^{\pi/2} = -\sqrt{2}\left[\sin\left(-\frac{\pi}{4}\right)-\sin\frac{\pi}{4}\right]$$

$$= -\sqrt{2}\left[-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right] = \sqrt{2}\,.\frac{2}{\sqrt{2}} = 2.$$

Example 4. *Evaluate the following integrals :*

(i) $\displaystyle\int_0^{\pi/4} \sqrt{1-\sin 2x}\; dx$ (ii) $\displaystyle\int_{\pi}^{3\pi/2} \sqrt{1-\cos 2x}\; dx.$

Solution. (*i*) $\displaystyle\int_0^{\pi/4} \sqrt{1-\sin 2x}\; dx = \int_0^{\pi/4} \sqrt{1-\cos\left(\frac{\pi}{2}-2x\right)}\, dx = \int_0^{\pi/4} \sqrt{2\sin^2\left(\frac{\pi}{4}-x\right)}\; dx$

$$= \sqrt{2}\int_0^{\pi/4} \left|\sin\left(\frac{\pi}{4}-x\right)\right| dx = \sqrt{2}\int_0^{\pi/4} \sin\left(\frac{\pi}{4}-x\right) dx$$

$\left[\text{As } 0 \le x \le \frac{\pi}{4} \Rightarrow 0 \ge -x \ge -\frac{\pi}{4} \Rightarrow \frac{\pi}{4} \ge \frac{\pi}{4}-x \ge 0\right.$

$\left.\Rightarrow 0 \le \frac{\pi}{4}-x \le \frac{\pi}{4} \Rightarrow \sin\left(\frac{\pi}{4}-x\right) \ge 0 \Rightarrow \left|\sin\left(\frac{\pi}{4}-x\right)\right| = \sin\left(\frac{\pi}{4}-x\right)\right]$

$$= \sqrt{2}\left[\frac{-\cos\left(\frac{\pi}{4}-x\right)}{-1}\right]_0^{\pi/4} = \sqrt{2}\left[\cos 0-\cos\frac{\pi}{4}\right]$$

$$= \sqrt{2}\left(1-\frac{1}{\sqrt{2}}\right) = \sqrt{2}-1.$$

(*ii*) $\displaystyle\int_{\pi}^{3\pi/2} \sqrt{1-\cos 2x}\; dx = \int_{\pi}^{3\pi/2} \sqrt{2\sin^2 x}\; dx = \sqrt{2}\int_{\pi}^{3\pi/2} |\sin x|\; dx$

$$= \sqrt{2}\int_{\pi}^{3\pi/2} (-\sin x)\; dx$$

$\left[\text{As } \pi \le x \le \frac{3\pi}{2} \Rightarrow \sin x \le 0 \Rightarrow |\sin x| = -\sin x\right]$

$$= -\sqrt{2}\left[-\cos x\right]_{\pi}^{3\pi/2} = \sqrt{2}\left(\cos\frac{3\pi}{2}-\cos\pi\right)$$

$$= \sqrt{2}(0-(-1)) = \sqrt{2}.$$

Example 5. *Evaluate the following integrals :*

(i) $\displaystyle\int_{\pi/4}^{\pi/2} \sqrt{1-\sin 2x}\; dx$ (ii) $\displaystyle\int_0^{2\pi} \sqrt{1+\sin\frac{x}{2}}\; dx\,.$

Solution. (*i*) $\displaystyle\int_{\pi/4}^{\pi/2} \sqrt{1-\sin 2x}\; dx = \int_{\pi/4}^{\pi/2} \sqrt{1-\cos\left(\frac{\pi}{2}-2x\right)}\; dx = \int_{\pi/4}^{\pi/2} \sqrt{2\sin^2\left(\frac{\pi}{4}-x\right)}\, dx$

$$= \sqrt{2}\int_{\pi/4}^{\pi/2} \left|\sin\left(\frac{\pi}{4}-x\right)\right| dx = \sqrt{2}\int_{\pi/4}^{\pi/2} \left(-\sin\left(\frac{\pi}{4}-x\right)\right) dx$$

$\left[\text{As } \frac{\pi}{4} \le x \le \frac{\pi}{2} \Rightarrow -\frac{\pi}{4} \ge -x \ge -\frac{\pi}{2} \Rightarrow 0 \ge \frac{\pi}{4}-x \ge -\frac{\pi}{4}\right.$

$\left.\Rightarrow -\frac{\pi}{4} \le \frac{\pi}{4}-x \le 0 \Rightarrow \sin\left(\frac{\pi}{4}-x\right) \le 0 \Rightarrow \left|\sin\left(\frac{\pi}{4}-x\right)\right| = -\sin\left(\frac{\pi}{4}-x\right)\right]$

$$= -\sqrt{2}\left[\frac{-\cos\left(\frac{\pi}{4}-x\right)}{-1}\right]_{\pi/4}^{\pi/2} = -\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)-\cos 0\right)$$

$$= -\sqrt{2}\left(\frac{1}{\sqrt{2}}-1\right) = \sqrt{2}-1.$$

(*ii*) $$\int_0^{2\pi}\sqrt{1+\sin\frac{x}{2}}\,dx = \int_0^{2\pi}\sqrt{1+\cos\left(\frac{\pi}{2}-\frac{x}{2}\right)}\,dx = \int_0^{2\pi}\sqrt{2\cos^2\left(\frac{\pi}{4}-\frac{x}{4}\right)}\,dx$$

$$= \sqrt{2}\int_0^{2\pi}\left|\cos\left(\frac{\pi}{4}-\frac{x}{4}\right)\right|dx = \sqrt{2}\int_0^{2\pi}\cos\left(\frac{\pi}{4}-\frac{x}{4}\right)dx$$

$$\left[\text{As } 0 \le x \le 2\pi \Rightarrow 0 \le \frac{x}{4} \le \frac{\pi}{2} \Rightarrow 0 \ge -\frac{x}{4} \ge -\frac{\pi}{2} \Rightarrow \frac{\pi}{4} \ge \frac{\pi}{4}-\frac{x}{4} \ge -\frac{\pi}{4}\right.$$

$$\left.\Rightarrow -\frac{\pi}{4} \le \frac{\pi}{4}-\frac{x}{4} \le \frac{\pi}{4} \Rightarrow \cos\left(\frac{\pi}{4}-\frac{x}{4}\right) > 0 \Rightarrow \left|\cos\left(\frac{\pi}{4}-\frac{x}{4}\right)\right| = \cos\left(\frac{\pi}{4}-\frac{x}{4}\right)\right]$$

$$= \sqrt{2}\left[\frac{\sin\left(\frac{\pi}{4}-\frac{x}{4}\right)}{-\frac{1}{4}}\right]_0^{2\pi} = -4\sqrt{2}\left(\sin\left(-\frac{\pi}{4}\right)-\sin\frac{\pi}{4}\right)$$

$$= -4\sqrt{2}\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right) = 4\sqrt{2}\cdot\frac{2}{\sqrt{2}} = 8.$$

Example 6. *Evaluate the following integrals :*

(*i*) $\displaystyle\int_0^{\pi/4} \sec x\sqrt{\frac{1-\sin x}{1+\sin x}}\,dx$ (*ii*) $\displaystyle\int_0^{\pi/4}(\tan x+\cot x)^{-2}\,dx.$

Solution. (*i*) $$\int_0^{\pi/4}\sec x\sqrt{\frac{1-\sin x}{1+\sin x}}\,dx = \int_0^{\pi/4}\sec x\sqrt{\frac{1-\sin x}{1+\sin x}\times\frac{1-\sin x}{1-\sin x}}\,dx$$

$$= \int_0^{\pi/4}\sec x\sqrt{\frac{(1-\sin x)^2}{\cos^2 x}}\,dx = \int_0^{\pi/4}\sec x\left|\frac{1-\sin x}{\cos x}\right|dx$$

$$\left[\text{As } 0 \le x \le \frac{\pi}{4} \Rightarrow \cos x > 0,\ 1-\sin x > 0 \Rightarrow \frac{1-\sin x}{\cos x} > 0\right]$$

$$= \int_0^{\pi/4}\sec x\,.\,\frac{1-\sin x}{\cos x}\,dx = \int_0^{\pi/4}\sec x\,(\sec x-\tan x)\,dx$$

$$= \int_0^{\pi/4}(\sec^2 x-\sec x\tan x)\,dx = \Big[\tan x-\sec x\Big]_0^{\pi/4}$$

$$= \left(\tan\frac{\pi}{4}-\sec\frac{\pi}{4}\right)-(\tan 0-\sec 0)$$

$$= (1-\sqrt{2})-(0-1) = 2-\sqrt{2}.$$

(*ii*) $$\int_0^{\pi/4}(\tan x+\cot x)^{-2}\,dx = \int_0^{\pi/4}\left(\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}\right)^{-2}dx = \int_0^{\pi/4}\left(\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}\right)^{-2}dx$$

$$= \int_0^{\pi/4}\left(\frac{1}{\sin x\cos x}\right)^{-2}dx = \int_0^{\pi/4}(\sin x\cos x)^2\,dx$$

$$= \frac{1}{4}\int_0^{\pi/4}(2\sin x\cos x)^2\,dx = \frac{1}{4}\int_0^{\pi/4}\sin^2 2x\,dx$$

$$= \frac{1}{4}\int_0^{\pi/4}\frac{1-\cos 4x}{2}\,dx = \frac{1}{8}\left[x - \frac{\sin 4x}{4}\right]_0^{\pi/4}$$

$$= \frac{1}{8}\left[\left(\frac{\pi}{4} - \frac{\sin\pi}{4}\right) - \left(0 - \frac{\sin 0}{4}\right)\right] = \frac{1}{8}\left[\left(\frac{\pi}{4} - 0\right) - 0\right] = \frac{\pi}{32}.$$

Example 7. *Evaluate the following integrals :*

(*i*) $\int_0^1 \frac{(\sin^{-1} x)^2}{\sqrt{1-x^2}}\,dx$ (*ii*) $\int_0^1 \frac{x+1}{(x^2+2x+3)^2}\,dx.$

Solution. (*i*) $\int_0^1 \frac{(\sin^{-1} x)^2}{\sqrt{1-x^2}}\,dx = \int_0^1 (\sin^{-1} x)^2 . \frac{1}{\sqrt{1-x^2}}\,dx$ $\quad\Big|\int (f(x))^n f'(x)\,dx$ form

$$= \left[\frac{(\sin^{-1} x)^3}{3}\right]_0^1 = \frac{1}{3}\left[(\sin^{-1} 1)^3 - (\sin^{-1} 0)^3\right]$$

$$= \frac{1}{3}\left[\left(\frac{\pi}{2}\right)^3 - 0^3\right] = \frac{\pi^3}{24}.$$

(*ii*) $\int_0^1 \frac{x+1}{(x^2+2x+3)^2}\,dx = \frac{1}{2}\int_0^1 (x^2+2x+3)^{-2}(2x+2)\,dx$

$$= \frac{1}{2}\cdot\left[\frac{(x^2+2x+3)^{-1}}{-1}\right]_0^1 = -\frac{1}{2}\left[\frac{1}{x^2+2x+3}\right]_0^1$$

$$= -\frac{1}{2}\left[\frac{1}{1+2+3} - \frac{1}{0+0+3}\right] = -\frac{1}{2}\left(\frac{1}{6} - \frac{1}{3}\right) = -\frac{1}{2}\left(-\frac{1}{6}\right) = \frac{1}{12}.$$

Example 8. *Evaluate the following integrals :*

(*i*) $\int_a^b \frac{\log x}{x}\,dx$ (*I.S.C. 2000*) (*ii*) $\int_0^{\pi/4} \frac{2\cos 2x}{1+\sin 2x}\,dx$. (*I.S.C. 2002*)

Solution. (*i*) $\int_a^b \frac{\log x}{x}\,dx = \int_a^b (\log x)^1 . \frac{1}{x}\,dx$ $\quad\Big|\int (f(x))^n f'(x)\,dx$ form

$$= \left[\frac{(\log x)^2}{2}\right]_a^b = \frac{1}{2}[(\log b)^2 - (\log a)^2]$$

$$= \frac{1}{2}(\log b + \log a)(\log b - \log a)$$

$$= \frac{1}{2}\log ab\,\log\left(\frac{b}{a}\right).$$

(*ii*) As $\frac{d}{dx}(1 + \sin 2x) = 0 + \cos 2x . 2 = 2\cos 2x,$

$\therefore \int_0^{\pi/4} \frac{2\cos 2x}{1+\sin 2x}\,dx = \left[\log|1+\sin 2x|\right]_0^{\pi/4}$ $\quad\Big|\int \frac{f'(x)}{f(x)}\,dx$ form

$$= \log|1 + \sin\frac{\pi}{2}| - \log|1 + \sin 0|$$

$$= \log(1 + 1) - \log(1 + 0) = \log 2 - \log 1$$

$$= \log 2 - 0 = \log 2.$$

Example 9. *Evaluate the following :*

(i) $\int_0^1 \frac{x^9}{5+x^{10}}\,dx$ (ii) $\int_0^{\pi/2} x \cos 2x\,dx.$

Solution. (*i*) $\int_0^1 \frac{x^9}{5+x^{10}}\,dx = \frac{1}{10}\int_0^1 \frac{10.x^9}{5+x^{10}}\,dx$ $\left|\int \frac{f'(x)}{f(x)}dx\right.$ form

$$= \frac{1}{10}\left[\log|5+x^{10}|\right]_0^1 = \frac{1}{10}[\log 6 - \log 5] = \frac{1}{10}\log\frac{6}{5}.$$

(*ii*) $\int_0^{\pi/2} x\cos 2x\,dx = \left[x.\frac{\sin 2x}{2}\right]_0^{\pi/2} - \int_0^{\pi/2} 1.\frac{\sin 2x}{2}\,dx$ (using integration by parts)

$$= \frac{1}{2}\left(\frac{\pi}{2}\sin\pi - 0\right) - \frac{1}{2}\int_0^{\pi/2}\sin 2x\,dx = \frac{1}{2}\left(\frac{\pi}{2}.0\right) - \frac{1}{2}\left[-\frac{\cos 2x}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{4}(\cos\pi - \cos 0) = \frac{1}{4}(-1-1) = -\frac{1}{2}.$$

Example 10. *Evaluate the following :*

(i) $\int_0^{\pi/2} x\sin^2 x\,dx$ (ii) $\int_0^1 x^2 e^x\,dx.$

Solution. (*i*) $\int_0^{\pi/2} x\sin^2 x\,dx = \int_0^{\pi/2} x.\frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\int_0^{\pi/2} x\,dx - \frac{1}{2}\int_0^{\pi/2} x\cos 2x\,dx$

$$= \frac{1}{2}.\left[\frac{x^2}{2}\right]_0^{\pi/2} - \frac{1}{2}\left(\left[x.\frac{\sin 2x}{2}\right]_0^{\pi/2} - \int_0^{\pi/2} 1.\frac{\sin 2x}{2}dx\right)$$

$$= \frac{1}{4}\left(\frac{\pi^2}{4}-0\right) - \frac{1}{4}\left(\frac{\pi}{2}\sin\pi - 0\right) + \frac{1}{4}\int_0^{\pi/2}\sin 2x\,dx$$

$$= \frac{\pi^2}{16} - \frac{1}{4}\left(\frac{\pi}{2}.0\right) + \frac{1}{4}\left[\frac{-\cos 2x}{2}\right]_0^{\pi/2} = \frac{\pi^2}{16} - \frac{1}{8}\left[\cos\ 2x\right]_0^{\pi/2}$$

$$= \frac{\pi^2}{16} - \frac{1}{8}(\cos\pi - \cos 0) = \frac{\pi^2}{16} - \frac{1}{8}(-1-1) = \frac{\pi^2}{16} + \frac{1}{4}.$$

(*ii*) $\int_0^1 x^2 e^x\,dx = \left[x^2.e^x\right]_0^1 - \int_0^1 2x\,e^x\,dx$

$$= (1.e^1 - 0) - 2\int_0^1 x\,e^x\,dx = e - 2\left(\left[x\,e^x\right]_0^1 - \int_0^1 1.e^x\,dx\right)$$

$$= e - 2(1.e^1 - 0) + 2\int_0^1 e^x\,dx = e - 2e + 2\left[e^x\right]_0^1$$

$$= -e + 2(e^1 - e^0) = -e + 2(e-1) = e - 2.$$

Example 11. *Evaluate the following integrals :*

(i) $\int_{\pi/4}^{\pi/2}\cos 2x\log\sin x\,dx$ (ii) $\int_0^1 x\tan^{-1}x\,dx.$ (*I.S.C. 2002*)

Solution. (*i*) $\int_{\pi/4}^{\pi/2}\cos 2x\log\sin x\,dx = \int_{\pi/4}^{\pi/2}\log\sin x.\cos 2x\,dx$ (integrate by parts)

$$= \left[\log\sin x.\frac{\sin 2x}{2}\right]_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2}\frac{1}{\sin x}.\cos x.\frac{\sin 2x}{2}\,dx$$

$$= \frac{1}{2}\left[0 - \log\frac{1}{\sqrt{2}}\right] - \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x} \cdot \frac{2\sin x \cos x}{2}\, dx$$

$$= -\frac{1}{2}\left(\log 1 - \frac{1}{2}\log 2\right) - \int_{\pi/4}^{\pi/2} \cos^2 x\, dx$$

$$= \frac{1}{4}\log 2 - \int_{\pi/4}^{\pi/2} \frac{1 + \cos 2x}{2}\, dx$$

$$= \frac{1}{4}\log 2 - \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_{\pi/4}^{\pi/2}$$

$$= \frac{1}{4}\log 2 - \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{\pi}{4}\right) + \frac{1}{2}(0 - 1)\right] = \frac{1}{4}\log 2 - \frac{\pi}{8} + \frac{1}{4}.$$

(*ii*) $\int_0^1 x \tan^{-1} x\, dx = \int_0^1 \tan^{-1} x \,.\, x\, dx$ (integrate by parts)

$$= \left[\tan^{-1} x \,.\, \frac{x^2}{2}\right]_0^1 - \int_0^1 \frac{1}{1 + x^2} \cdot \frac{x^2}{2}\, dx$$

$$= \frac{1}{2}\left[\tan^{-1} 1 - 0\right] - \frac{1}{2}\int_0^1 \frac{x^2}{1 + x^2}\, dx$$

$$= \frac{1}{2} \cdot \frac{\pi}{4} - \frac{1}{2}\int_0^1 \left(1 - \frac{1}{1 + x^2}\right) dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\left[x - \tan^{-1} x\right]_0^1 = \frac{\pi}{8} - \frac{1}{2}\left[(1 - \tan^{-1} 1) - (0 - \tan^{-1} 0)\right]$$

$$= \frac{\pi}{8} - \frac{1}{2}\left[\left(1 - \frac{\pi}{4}\right) - 0\right] = \frac{\pi}{4} - \frac{1}{2}.$$

Example 12. *Evaluate the following :*

(*i*) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1 + \cos x + \sin x}\, dx$ (*ii*) $\displaystyle\int_1^5 \frac{\log x}{(x + 1)^2}\, dx.$

Solution. (*i*) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1 + \cos x + \sin x}\, dx = \int_0^{\pi/2} \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2} + 2\sin \frac{x}{2} \cos \frac{x}{2}}\, dx$

$$= \int_0^{\pi/2} \frac{\cos \frac{x}{2} - \sin \frac{x}{2}}{2\cos \frac{x}{2}}\, dx = \frac{1}{2}\int_0^{\pi/2} \left(1 - \tan \frac{x}{2}\right) dx$$

$$= \frac{1}{2}\left[x + \frac{\log\left|\cos \frac{x}{2}\right|}{\frac{1}{2}}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - 0\right) + 2\left(\log\frac{1}{\sqrt{2}} - \log 1\right)\right]$$

$$= \frac{\pi}{4} + \log 1 - \log \sqrt{2} = \frac{\pi}{4} - \frac{1}{2}\log 2.$$

$$(ii)\ \int_1^5 \frac{\log x}{(x+1)^2}\,dx = \int_1^5 \log x\,.\,(x+1)^{-2}\,dx \qquad \text{(integrate by parts)}$$

$$= \left[\log x\,.\frac{(x+1)^{-1}}{-1}\right]_1^5 - \int_1^5 \frac{1}{x}.\frac{(x+1)^{-1}}{-1}\,dx$$

$$= -\left[\frac{\log x}{x+1}\right]_1^5 + \int_1^5 \frac{dx}{x(x+1)}$$

$$= -\left(\frac{\log 5}{6} - \frac{\log 1}{2}\right) + \int_1^5 \left(\frac{1}{x} - \frac{1}{x+1}\right) dx \qquad \text{(by partial fractions)}$$

$$= -\left(\frac{\log 5}{6} - 0\right) + \left[\log|x| - \log|x+1|\right]_1^5$$

$$= -\frac{1}{6}\log 5 + (\log 5 - \log 6) - (\log 1 - \log 2)$$

$$= \frac{5}{6}\log 5 - (\log 6 - \log 2) = \frac{5}{6}\log 5 - \log 3.$$

Example 13. *Evaluate the following :*

$$(i)\ \int_0^{\sqrt{2}} \sqrt{2-x^2}\,dx \qquad (ii)\ \int_1^2 \frac{2}{4x^2-1}\,dx.$$

Solution. (i) $\displaystyle\int_0^{\sqrt{2}} \sqrt{2-x^2}\,dx = \int_0^{\sqrt{2}} \sqrt{\left(\sqrt{2}\right)^2 - x^2}\,dx$

$$= \left[\frac{x\sqrt{2-x^2}}{2} + \frac{\left(\sqrt{2}\right)^2}{2}\sin^{-1}\frac{x}{\sqrt{2}}\right]_0^{\sqrt{2}}$$

$$= \left(\frac{\sqrt{2}.0}{2} + \sin^{-1}1\right) - \left(\frac{0\sqrt{2}}{2} + \sin^{-1}0\right)$$

$$= \left(0 + \frac{\pi}{2}\right) - (0+0) = \frac{\pi}{2}.$$

$$(ii)\ \int_1^2 \frac{2}{4x^2-1}\,dx = 2.\frac{1}{4}\int_1^2 \frac{dx}{x^2 - \left(\frac{1}{2}\right)^2} = \frac{1}{2}.\frac{1}{2.\frac{1}{2}}\left[\log\left|\frac{x-\frac{1}{2}}{x+\frac{1}{2}}\right|\right]_1^2 = \frac{1}{2}\left[\log\left|\frac{2x-1}{2x+1}\right|\right]_1^2$$

$$= \frac{1}{2}\left(\log\frac{3}{5} - \log\frac{1}{3}\right) = \frac{1}{2}\log\left(\frac{3}{5}\times\frac{3}{1}\right) = \frac{1}{2}\log\frac{9}{5}.$$

Example 14. *Evaluate the following integrals :*

$$(i)\ \int_0^1 \frac{x e^x}{(x+1)^2}\,dx \quad \textit{(I.S.C. 2011)} \qquad (ii)\ \int_1^5 \frac{\log x}{(x+1)^2}\,dx \qquad (iii)\ \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx.$$

Solution. (i) $\displaystyle\int_0^1 \frac{x e^x}{(x+1)^2}\,dx = \int_0^1 \frac{(x+1)-1}{(x+1)^2}\,e^x\,dx$

$$= \int_0^1 \frac{1}{x+1}.\,e^x\,dx - \int_0^1 \frac{1}{(x+1)^2}\,e^x\,dx$$

(evaluate the first integral by parts, taking $\frac{1}{x+1}$ as the first function)

$$= \left[\frac{1}{x+1}\cdot e^x\right]_0^1 - \int_0^1 (-1)(x+1)^{-2}\cdot e^x\,dx - \int_0^1 \frac{1}{(x+1)^2}\,e^x\,dx$$

$$= \left(\frac{1}{2}\cdot e^1 - 1\cdot e^0\right) + \int_0^1 \frac{1}{(x+1)^2}\,e^x\,dx - \int_0^1 \frac{1}{(x+1)^2}\,e^x\,dx = \frac{1}{2}\,e - 1.$$

(*ii*) $$\int_1^5 \frac{\log x}{(x+1)^2}\,dx = \int_1^5 \log x\,.\,(x+1)^{-2}\,dx \qquad \text{(by parts)}$$

$$= \left[\log x\,.\,\frac{(x+1)^{-1}}{-1}\right]_1^5 - \int_1^5 \frac{1}{x}\cdot\frac{(x+1)^{-1}}{-1}\,dx$$

$$= -\left[\frac{\log x}{x+1}\right]_1^5 + \int_1^5 \frac{1}{x(x+1)}\,dx$$

$$= -\left[\frac{\log 5}{6} - \frac{\log 1}{2}\right] + \int_1^5 \left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$= -\left(\frac{\log 5}{6} - 0\right) + \Big[\log|x| - \log|x+1|\Big]_1^5$$

$$= -\frac{\log 5}{6} + (\log 5 - \log 6) - (\log 1 - \log 2)$$

$$= \frac{5}{6}\log 5 - (\log 6 - \log 2) = \frac{5}{6}\log 5 - \log 3.$$

(*iii*) $$\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int_0^1 2\tan^{-1}x\,dx = 2\int_0^1 \tan^{-1}x\,.\,1\,dx \qquad \text{(integrate by parts)}$$

$$= 2\left[\Big[\tan^{-1}x\,.\,x\Big]_0^1 - \int_0^1 \frac{1}{1+x^2}\,.\,x\,dx\right]$$

$$= 2(\tan^{-1}1 - 0) - \int_0^1 \frac{2x}{1+x^2}\,dx$$

$$= 2\left(\frac{\pi}{4} - 0\right) - \Big[\log(1+x^2)\Big]_0^1$$

$$= \frac{\pi}{2} - (\log 2 - \log 1) = \frac{\pi}{2} - (\log 2 - 0)$$

$$= \frac{\pi}{2} - \log 2.$$

Example 15. *Evaluate the following :*

(*i*) $\displaystyle\int_{1/4}^{1/2} \frac{dx}{\sqrt{x-x^2}}$ (*ii*) $\displaystyle\int_2^3 \frac{x^3+1}{x(x-1)}\,dx.$

Solution. (*i*) $$\int_{1/4}^{1/2} \frac{dx}{\sqrt{x-x^2}} = \int_{1/4}^{1/2} \frac{dx}{\sqrt{\frac{1}{4} - \left(x^2 - x + \frac{1}{4}\right)}} = \int_{1/4}^{1/2} \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}}$$

$$= \left[\sin^{-1}\left(\frac{x - \frac{1}{2}}{\frac{1}{2}}\right)\right]_{1/4}^{1/2} = \Big[\sin^{-1}(2x-1)\Big]_{1/4}^{1/2}$$

$$= \sin^{-1}0 - \sin^{-1}\left(-\frac{1}{2}\right) = 0 - \left(-\frac{\pi}{6}\right) = \frac{\pi}{6}.$$

(*ii*) $\int_2^3 \frac{x^3+1}{x(x-1)}\,dx = \int_2^3 \left(x+1+\frac{x+1}{x(x-1)}\right) dx$ (by division)

Let $\frac{x+1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \Rightarrow x + 1 = A(x-1) + Bx.$

On putting $x = 0$ and $x = 1$, we get

$1 = -A$ and $2 = B \quad \Rightarrow A = -1$ and $B = 2.$

$$\therefore \int_2^3 \frac{x^3+1}{x(x-1)}\,dx = \int_2^3 \left(x+1-\frac{1}{x}+\frac{2}{x-1}\right) dx$$

$$= \left[\frac{x^2}{2}+x-\log|x|+2\log|x-1|\right]_2^3$$

$$= \left(\frac{9}{2}+3-\log 3+2\log 2\right) - (2+2-\log 2-2\log 1)$$

$$= \frac{15}{2} - \log 3 + 2\log 2 - 4 + \log 2 - 2.0 = \frac{7}{2} + 3\log 2 - \log 3.$$

Example 16. *Prove that :* $\int_0^{\pi/2} \frac{3\sin\theta+4\cos\theta}{\sin\theta+\cos\theta}\,d\theta = \frac{7\pi}{4}.$ (*I.S.C. 2005*)

Solution. Let $3\sin\theta + 4\cos\theta = l(\sin\theta+\cos\theta) + m(\cos\theta - \sin\theta).$

Equating coefficients of $\sin\theta$ and $\cos\theta$ on both sides, we get

$$3 = l - m \text{ and } 4 = l + m.$$

Solving these for l and m, we get $l = \frac{7}{2}$ and $m = \frac{1}{2}$.

$$\therefore \int_0^{\pi/2} \frac{3\sin\theta+4\cos\theta}{\sin\theta+\cos\theta}\,d\theta = \int_0^{\pi/2} \frac{\frac{7}{2}(\sin\theta+\cos\theta)+\frac{1}{2}(\cos\theta-\sin\theta)}{\sin\theta+\cos\theta}\,d\theta$$

$$= \int_0^{\pi/2} \left(\frac{7}{2}+\frac{1}{2}\cdot\frac{\cos\theta-\sin\theta}{\sin\theta+\cos\theta}\right) d\theta$$

$$= \frac{7}{2}[\theta]_0^{\pi/2} + \frac{1}{2}\left[\log|\sin\theta+\cos\theta|\right]_0^{\pi/2}$$

$$= \frac{7}{2}\left[\frac{\pi}{2}-0\right] + \frac{1}{2}[\log 1 - \log 1] = \frac{7\pi}{4}.$$

EXERCISE 10.1

Evaluate the following (1 to 21) definite integrals :

1. (*i*) $\int_0^8 \left(\sqrt{8x}-\frac{x^2}{8}\right) dx$ (*ii*) $\int_0^1 \frac{1}{2x-3}\,dx.$

2. (*i*) $\int_{-4}^{-1} \frac{1}{x}\,dx$ (*ii*) $\int_{\pi/6}^{\pi/4} \operatorname{cosec} x\,dx.$

3. (*i*) $\int_0^1 \sqrt{5x+4}\,dx$ (*ii*) $\int_0^1 \frac{dx}{\sqrt{1+x}+\sqrt{x}}.$

4. (*i*) $\int_0^{\pi} \frac{dx}{1+\sin x}$ (*ii*) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}.$

5. (*i*) $\int_0^1 \frac{1-x}{1+x}\,dx$ (*ii*) $\int_0^{\pi/2} \frac{\cos x}{5+4\sin x}\,dx.$

6. (*i*) $\int_1^3 (x^2+e^x)\,(x^3+3e^x+4)\,dx$ (*ii*) $\int_1^2 \frac{3x}{9x^2-1}\,dx.$

7. (*i*) $\int_0^{\pi/4} \tan^3 x \sec^2 x\,dx$ (*ii*) $\int_0^{\pi/2} \sin^4 x\,dx.$

8. (*i*) $\int_0^{\pi/2} \sqrt{1-\cos 2x}\,dx$ (*ii*) $\int_0^{\pi/4} \sqrt{1+\sin 2x}\,dx.$

9. (*i*) $\int_0^{\pi/2} \frac{\sin^2 x}{(1+\cos x)^2}\,dx$ (*ii*) $\int_0^{\pi/2} \frac{\sin\theta}{\sqrt{1+\cos\theta}}\,d\theta.$

10. (*i*) $\int_0^{\pi/4} \sin 2x \sin 3x\,dx$ (*ii*) $\int_0^{\pi/2} (a^2\cos^2 x + b^2\sin^2 x)\,dx.$

11. (*i*) $\int_0^{\pi/4} 2\tan^3 x\,dx$ (*ii*) $\int_0^{\pi/4} \frac{\tan^3 x}{1+\cos 2x}\,dx.$

12. (*i*) $\int_0^{\pi/4} \frac{\sin x}{\cos 3x + 3\cos x}\,dx$ (*ii*) $\int_{\pi/3}^{\pi/4} (\tan x + \cot x)^2\,dx$

(*iii*) $\int_0^{\pi/4} (\tan x + \cot x)^{-1}\,dx.$ (*I.S.C. 2003*)

13. (*i*) $\int_0^4 \frac{dx}{\sqrt{x^2+2x+3}}$ (*ii*) $\int_0^a \frac{dx}{\sqrt{ax-x^2}}.$

14. (*i*) $\int_0^1 \frac{\tan^{-1} x}{1+x^2}\,dx$ (*ii*) $\int_0^1 \frac{x^5}{1+x^6}\,dx.$

15. (*i*) $\int_0^1 x\,e^x\,dx$ (*ii*) $\int_1^2 \frac{x+3}{x(x+2)}\,dx.$

16. (*i*) $\int_0^{\pi/2} x^2\cos 2x\,dx$ (*ii*) $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\,dx.$

17. (*i*) $\int_1^3 \frac{dx}{x^2(x+1)}$ (*ii*) $\int_1^2 \frac{dx}{(x+1)(x^2-7x+12)}.$

18. (*i*) $\int_0^2 \frac{1}{\sqrt{3+2x-x^2}}\,dx$ (*ii*) $\int_0^2 \frac{dx}{4+x-x^2}.$

19. (*i*) $\int_0^1 \sin^{-1} x\,dx$ (*ii*) $\int_0^1 \tan^{-1} x\,dx.$

20. (*i*) $\int_1^2 e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$ (*ii*) $\int_0^{\pi/2} \frac{x+\sin x}{1+\cos x}\,dx.$

21. (*i*) $\int_{\pi/4}^{\pi/2} e^x(\log(\sin x) + \cot x)\, dx$ (*ii*) $\int_{\pi/2}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right) dx.$

22. If $\int_0^a 3x^2\, dx = 8$, find the value of a.

10.1.2 Evaluation of Definite Integrals by Substitution

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate the following :*

(*i*) $\int_0^1 x^2 e^{x^3} dx$ (*ii*) $\int_0^1 \frac{x}{1+x^4} dx.$

Solution. (*i*) Put $x^3 = t \Rightarrow 3x^2\, dx = dt \Rightarrow x^2\, dx = \frac{1}{3} dt.$

When $x = 0$, $t = 0$ and when $x = 1$, $t = 1^3 = 1$.

$$\therefore \quad \int_0^1 x^2 e^{x^3} dx = \int_0^1 e^t \frac{1}{3} dt = \frac{1}{3}\left[e^t\right]_0^1$$

$$= \frac{1}{3}[e^1 - e^0] = \frac{1}{3}(e-1).$$

(*ii*) Put $x^2 = t \Rightarrow 2x\, dx = dt \Rightarrow x\, dx = \frac{1}{2} dt.$

When $x = 0$, $t = 0$ and when $x = 1$, $t = 1$.

$$\therefore \quad \int_0^1 \frac{x}{1+x^4} dx = \int_0^1 \frac{1}{1+t^2}\cdot\frac{1}{2} dt = \frac{1}{2}\cdot\frac{1}{1}\left[\tan^{-1}\frac{t}{1}\right]_0^1$$

$$= \frac{1}{2}[\tan^{-1} 1 - \tan^{-1} 0] = \frac{1}{2}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{8}.$$

Example 2. *Evaluate the following :*

(*i*) $\int_0^1 \frac{e^x}{1+e^{2x}} dx$ (*ii*) $\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x} dx$ (*iii*) $\int_0^{\pi/4} \frac{\sin^2 x \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx.$

Solution. (*i*) Put $e^x = t \Rightarrow e^x\, dx = dt$.

When $x = 0$, $t = e^0 = 1$ and when $x = 1$, $t = e^1 = e$.

$$\therefore \quad \int_0^1 \frac{e^x}{1+e^{2x}} dx = \int_1^e \frac{dt}{1+t^2} = \frac{1}{1}\left[\tan^{-1}\frac{t}{1}\right]_1^e$$

$$= \tan^{-1} e - \tan^{-1} 1 = \tan^{-1} e - \frac{\pi}{4}.$$

(*ii*) Put $\cos x = t \Rightarrow -\sin x\, dx = dt \Rightarrow \sin x\, dx = -dt$.

When $x = 0$, $t = \cos 0 = 1$ and when $x = \frac{\pi}{2}$, $t = \cos\frac{\pi}{2} = 0$.

$$\therefore \quad \int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x} dx = \int_1^0 \frac{-dt}{1+t^2} = -\frac{1}{1}\left[\tan^{-1}\frac{t}{1}\right]_1^0$$

$$= -[\tan^{-1} 0 - \tan^{-1} 1] = -\left(0 - \frac{\pi}{4}\right) = \frac{\pi}{4}.$$

15. (*i*) 1 (*ii*) $\frac{1}{2}\log 6$. **16.** (*i*) $-\frac{\pi}{4}$ (*ii*) $a\pi$.

17. (*i*) $\frac{2}{3}+\log\frac{2}{3}$ (*ii*) $\frac{2}{5}\log 2-\frac{3}{20}\log 3$.

18. (*i*) $\frac{\pi}{3}$ (*ii*) $\frac{1}{\sqrt{17}}\log\left(\frac{21+5\sqrt{17}}{4}\right)$.

19. (*i*) $\frac{\pi}{2}-1$ (*ii*) $\frac{\pi}{4}-\frac{1}{2}\log 2$. **20.** (*i*) $\frac{1}{2}e^2-e$ (*ii*) $\frac{\pi}{2}$.

21. (*i*) $\frac{1}{2}e^{\pi/4}\log 2$ (*ii*) $e^{\pi/2}$. **22.** 2.

EXERCISE 10.2

1. (*i*) $\frac{1}{2}(e-1)$ (*ii*) $\frac{1}{2}\tan^{-1}e^2-\frac{\pi}{2}$. **2.** (*i*) $2(\sqrt{2}-1)$ (*ii*) $\frac{8}{21}$.

3. (*i*) $\tan^{-1}\left(\frac{1}{3}\right)$ (*ii*) $\tan^{-1}\left(\frac{1}{3}\right)$. **4.** (*i*) $\sin(\log 3)$ (*ii*) $\frac{\log 2}{1+\log 2}$.

5. (*i*) $\frac{16}{15}(2+\sqrt{2})$ (*ii*) $\frac{132}{7}\sqrt[3]{4}$. **6.** (*i*) $1-\log 2$ (*ii*) $\frac{\pi}{2}-1$.

7. (*i*) $\log\frac{4}{3}$ (*ii*) $\log\frac{4}{3}$. **8.** (*i*) $\frac{\pi}{8}$ (*ii*) $\log\frac{2e-1}{e}$.

9. (*i*) $\frac{3}{16}\pi a^4$ (*ii*) $\frac{\pi}{2ab}$. **10.** $\frac{\pi}{2}-\log 2$ (*ii*) $\frac{\pi}{2}-\log 2$.

11. (*i*) $\frac{\pi^2}{16}-\frac{\pi}{4}+\frac{1}{2}\log 2$ (*ii*) $e-\frac{2}{\log 2}$.

12. (*i*) $\frac{2}{3}\tan^{-1}\frac{1}{3}$ (*ii*) $\frac{1}{\sqrt{5}}\log\left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right)$.

13. (*i*) $\frac{\pi}{3}$ (*ii*) $\frac{\pi}{\sqrt{35}}$. **14.** (*i*) $\frac{1}{2}\log 3$ (*ii*) $\frac{1}{24}$.

15. (*i*) $\frac{\pi}{4}$ (*ii*) $\frac{\pi}{2}$.

EXERCISE 10.3

1. (*i*) $\frac{13}{2}$ (*ii*) 34 (*iii*) $\frac{25}{2}$.

2. (*i*) $2-\sqrt{2}$ (*ii*) 2 (*iii*) 1.

3. (*i*) 4 (*ii*) $2(e-1)$ (*iii*) $2-\frac{2}{e}$.

4. (*i*) 0 (*ii*) 4. **5.** 37. **6.** 47.

7. (*i*) 0 (*ii*) 0 (*iii*) 0.

8. (*i*) $\frac{\pi}{2}$ (*ii*) $\frac{3\pi}{8}$ (*iii*) 0 (*iv*) 0.

9. (*i*) 0 (*ii*) 0 (*iii*) 0 (*iv*) 0.

10. (*i*) $\frac{1}{42}$ (*ii*) $\frac{4}{63}$ (*iii*) $\frac{144}{35}\sqrt{3}$ (*iv*) $\frac{16}{315}a^{9/2}$.

11. (*i*) $\frac{\pi}{4}$ (*ii*) $\frac{\pi}{4}$ (*iii*) $\frac{\pi}{4}$ (*iv*) $\frac{\pi}{4}$.

12. (*i*) $\frac{\pi}{4}$ (*ii*) $\frac{\pi}{4}$ (*iii*) $\frac{\pi}{4}$ (*iv*) $\frac{\pi}{4}$.

13. (*i*) $\frac{\pi^2}{4}$ (*ii*) $\frac{2\pi}{3}$. **14.** (*i*) $\frac{\pi}{5}$ (*ii*) $\frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$.