4

PRINCIPLE OF MATHEMATICAL INDUCTION

INTRODUCTION

In drawing mathematical or scientific conclusions, there are two basic processes of reasoning that are commonly used. These are **induction** and **deduction—induction** *is the process of reasoning from particular to general and* **deduction** *is the process of reasoning from general to particular.* In this chapter, we shall study *induction.* Induction begins by observations, and from observations we arrive at some *tentative conclusions,* called **conjectures**. A conjecture may be true or false. The *principle of mathematical induction* helps us in proving some of these conjectures which are true.

4.1 MATHEMATICAL STATEMENT

A statement involving mathematical relation or relations is called a ***mathematical statement****.*

Consider the statements :

1. 6 is an even natural number.

2. $(x + 2)$ is a factor of $x^2 - 3x + 2$.

3. Mumbai is the capital of Maharashtra.

4. Sum of first n natural numbers is $\frac{n(n+1)}{2}$.

5. If A, B are any sets, then $A \cap B = B \cap A$.

Clearly, statements 1, 2, 4 and 5 are mathematical statements.

Notation for Mathematical Statements

Consider the mathematical statements :

1. $n(n + 1)$ is divisible by 2.

2. $3^{2n} - 1$ is divisible by 8.

3. $n^2 - n + 41$ is a prime integer.

4. If a set S contains n distinct objects, then the number of subsets of S is 2^n.

All these statements are concerned with natural number 'n', which takes values 1, 2, 3, … such statements are usually denoted by P(n) or S(n) etc. By giving particular values to n, we get particular statement. For example, if the statement "$2^{3n} - 1$ is divisible by 7" is denoted by P(n), then P(2) is the statement "$2^{3 \times 2} - 1$ is divisible by 7".

ILLUSTRATIVE EXAMPLES

Example 1. *If P(n) is the statement "$n^3 + n$ is divisible by 3", show that P(3) is true but P(4) is not true.*

Solution. P(n) is the statement "$n^3 + n$ is divisible by 3",

$\therefore$ P(3) is $3^3 + 3$ is divisible by 3 *i.e.* 30 is divisible by 3. Clearly it is true.

P(4) is $4^3 + 4$ is divisible by 3 *i.e.* 68 is divisible by 3. Clearly it is not true.

Example 2. *Let P(n) be the statement "$3^{2n} - 1$ is divisible by 8". What is P(n + 1) ?*

Solution. Given P(n) is the statement "$3^{2n} - 1$ is divisible by 8". To obtain P(n + 1), replace n by (n + 1) in P(n),

$\therefore$ P(n + 1) is the statement "$3^{2(n+1)} - 1$ is divisible by 8".

Example 3. *Let P(n) be the statement "$n^2 + n$ is an odd integer". Show that if P(m) is true then P(m + 1) is also true.*

Solution. Given statement P(n) is "$n^2 + n$ is an odd integer".

Let P(m) be true $\Rightarrow m^2 + m$ is an odd integer ...(i)

Now, $(m + 1)^2 + (m + 1) = m^2 + 2m + 1 + m + 1$

$= (m^2 + m) + (2m + 2)$

= an odd integer + $2(m + 1)$ (using (i))

= an odd integer + an even integer

($\because$ 2 divides $2(m + 1)$ for all $m \in \mathbf{N}$)

= an odd integer

$\Rightarrow$ P(m + 1) is true.

Example 4. *Let P(n) be the statement "$2^{3n} - 1$ is divisible by 7". Prove that if P(m) is true then P(m + 1) is also true.*

Solution. Given statement P(n) is "$2^{3n} - 1$ is divisible by 7".

Let P(m) be true $\Rightarrow 2^{3m} - 1$ is divisible by 7

$\Rightarrow 2^{3m} - 1 = 7\lambda$, for some integer λ

$\Rightarrow 2^{3m} = 1 + 7\lambda$ ...(i)

Now, $2^{3(m+1)} - 1 = 2^{3m}.2^3 - 1 = (1 + 7\lambda).8 - 1$ (using (i))

$= 7 + 56\lambda = 7(1 + 8\lambda)$, which is divisible by 7

$\Rightarrow$ P(m + 1) is true.

Example 5. *Let P(n) be the statement "$3^n > n$", show that if P(m) is true then P(m + 1) is also true.*

Solution. Given statement P(n) is "$3^n > n$".

Let P(m) be true

$\Rightarrow 3^m > m$

$\Rightarrow 3.3^m > 3.m$ (multiply both sides by 3)

$\Rightarrow 3^{m+1} > m + 2m$

$\Rightarrow 3^{m+1} > m + 1$ ($\because 2m > 1$ for every $m \in \mathbf{N}$)

$\Rightarrow$ P(m + 1) is true.

EXERCISE 4.1

1. If P(n) is the statement "$n(n + 1)(n + 2)$ is divisible by 6", then what is P(3) ?
   [**Ans.** 60 is divisible by 6]
2. If P (n) is the statement "$10n + 3$ is prime", then show that P (1) and P (2) are true but P (3) is not true.
3. If P(n) is the statement "$n(n + 1)(n + 2)$ is an integral multiple of 12", prove that P(3) and P(4) are true but P(5) is not true.
4. If P(n) is the statement "$n^2 - n + 41$ is prime", show that P(1), P(2), and P(3) are true but P(41) is not true.
5. Let P (n) be the statement "$n^2 + n$ is an even integer". Show that if P (k) is true then P (k + 1) is also true.

6. Let P(n) denote the statement "$3^{2n} - 1$ is a multiple of 8". Show that
(i) P(1), P(2) are true (ii) if P(m) is true then P(m + 1) is also true.

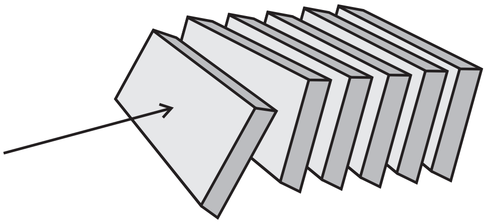
7. If P(n) is the statement "$n^2 > 100$", then show that whenever P (m) is true, P(m + 1) is also true.

8. Let P(n) be the statement "$2^n \geq 3n$". Show that if P(m) is true, then P(m + 1) is also true.

4.2 PRINCIPLE OF MATHEMATICAL INDUCTION

4.2.1 Motivation

To understand the basic principle of mathematical induction, suppose a set of thin rectangular tiles are placed as shown in the figure given below.



When the first tile is pushed in the indicated direction, all the tiles will fall. To be absolutely sure that all the tiles will fall, it is sufficient to know that

(i) the first tile falls, and

(ii) in the event that any tile falls its successor necessarily falls.

This is the underlying principle of mathematical induction.

A set S is said to be an inductive set if I $\in$ S and $x + 1 \in$ S whenever $x \in$ S.

We know that the set of natural number's **N** is a special ordered subset of the set of real numbers and it is the *smallest* subset of **R** which is inductive.

4.2.2 Principle of Mathematical Induction

Let P(n) be a statement involving the natural number n, then P(n) is true for all natural numbers n if

(i) P(1) is true

(ii) P(m + 1) is true whenever P(m) is true.

In other words, to prove that a statement P(n) is true for all natural numbers n, we have to go through two steps :

(i) verify the result for n = 1.

(ii) assume the result to be true for n = m and prove the result for n = m + 1.

REMARK

We emphasize that the proof by mathematical induction requires the fulfilment of both the conditions (i) and (ii) as stated above.

ILLUSTRATIVE EXAMPLES

Example 1. *Prove by mathematical induction that the sum of first n odd natural numbers is* n^2.

Solution. Let P(n) be the statement

"1 + 3 + 5 + … to n terms = n^2"

i.e. $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

($\because$ 1, 3, 5, … form an A.P., $\therefore$ $T_n = 1 + (n - 1).2 = 2n - 1$)

Now P(1) means $1 = 1^2$ *i.e.* 1 = 1, which is true

$\Rightarrow$ P(1) is true.

Let P(m) be true *i.e.* $1 + 3 + 5 + \ldots + (2m - 1) = m^2$ …(i)

For $P(m+1) : 1 + 3 + 5 + \ldots$ to $(m + 1)$ terms

$$= 1 + 3 + 5 + \ldots + (2(m + 1) - 1)$$
$$= 1 + 3 + 5 + \ldots + (2m + 1)$$
$$= [1 + 3 + 5 + \ldots + (2m - 1)] + (2m + 1)$$
$$= m^2 + (2m + 1) \qquad \text{(using } (i))$$
$$= m^2 + 2m + 1 = (m + 1)^2$$

$\Rightarrow$ $P(m + 1)$ is true.

Hence, by principle of mathematical induction, $P(n)$ is true for all n. Therefore, the sum of first n odd natural numbers is n^2.

Example 2. *Prove by induction that*

$1 + 5 + 9 + \ldots + (4n - 3) = n(2n - 1)$, *for all* $n \in \mathbf{N}$. (*NCERT Examplar Problems*)

Solution. Let $P(n)$ be the statement

$$1 + 5 + 9 + \ldots + (4n - 3) = n(2n - 1).$$

Now $P(1)$ means $1 = 1(2 . 1 - 1)$ *i.e.* $1 = 1$, which is true $\Rightarrow P(1)$ is true.

Let $P(m)$ be true

i.e. $1 + 5 + 9 + \ldots + (4m - 3) = m(2m - 1)$ ...(*i*)

For $P(m + 1) : 1 + 5 + 9 + \ldots + (4m - 3) + (4(m + 1) - 3)$

$$= m(2m - 1) + (4m + 1) \qquad \text{(using } (i))$$
$$= 2m^2 + 3m + 1 = (m + 1)(2m + 1)$$
$$= (m + 1)(2(m + 1) - 1)$$

$\Rightarrow$ $P(m + 1)$ is true.

Hence, by induction, $P(n)$ is true for all $n \in \mathbf{N}$.

Example 3. *Prove by principle of mathematical induction that*

$1.2 + 2.3 + 3.4 + \ldots + n(n + 1) = \dfrac{n(n + 1)(n + 2)}{3}$, *for all* $n \in \mathbf{N}$. (*NCERT*)

Solution. Let $P(n)$ be the statement

$$1.2 + 2.3 + 3.4 + \ldots + n(n + 1) = \frac{n(n + 1)(n + 2)}{3}.$$

Now $P(1)$ means $1.2 = \dfrac{1.2.3}{3}$ *i.e.* $2 = 2$, which is true

$\Rightarrow$ $P(1)$ is true.

Let $P(m)$ be true

i.e. $1.2 + 2.3 + 3.4 + \ldots + m(m + 1) = \dfrac{m(m + 1)(m + 2)}{3}$ ...(*i*)

For $P(m + 1) : 1.2 + 2.3 + 3.4 + \ldots + (m + 1)(\overline{m + 1} + 1)$

$$= 1.2 + 2.3 + 3.4 + \ldots + (m + 1)(m + 2)$$
$$= (1.2 + 2.3 + 3.4 + \ldots + m(m + 1)) + (m + 1)(m + 2)$$
$$= \frac{m(m + 1)(m + 2)}{3} + (m + 1)(m + 2) \qquad \text{(using } (i))$$
$$= (m + 1)(m + 2)\left(\frac{m}{3} + 1\right) = \frac{(m + 1)(m + 2)(m + 3)}{3}$$

$\Rightarrow$ $P(m + 1)$ is true.

Hence, by induction, $P(n)$ is true for all $n \in \mathbf{N}$.

Example 4. *For all $n \in \mathbf{N}$, prove by induction that*

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}. \qquad \text{(NCERT)}$$

Solution. Let P(n) be the statement $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Now P(1) means $1^2 = \frac{1(1+1)(2.1+1)}{6}$ *i.e.* $1 = \frac{2 \times 3}{6}$ *i.e.* $1 = 1$, which is true

$\Rightarrow$ P(1) is true.

Let P(m) be true

i.e. $1^2 + 2^2 + 3^2 + \ldots + m^2 = \frac{m(m+1)(2m+1)}{6}$ ...(i)

For P(m + 1) : $1^2 + 2^2 + 3^2 + \ldots + m^2 + (m+1)^2$

$$= (1^2 + 2^2 + 3^2 + \ldots + m^2) + (m+1)^2$$

$$= \frac{m(m+1)(2m+1)}{6} + (m+1)^2 \qquad \text{(using } (i))$$

$$= \frac{m(m+1)(2m+1) + 6(m+1)^2}{6} = \frac{(m+1)(2m^2 + m + 6m + 6)}{6}$$

$$= \frac{(m+1)(2m^2 + 7m + 6)}{6} = \frac{(m+1)(m+2)(2m+3)}{6}$$

$$= \frac{(m+1)(m+2)(2(m+1)+1)}{6}$$

$\Rightarrow$ P(m + 1) is true.

Hence, by induction, P(n) is true for all $n \in \mathbf{N}$.

Example 5. *Prove by induction that*

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2, \text{ for all } n \in \mathbf{N}. \qquad \text{(NCERT)}$$

Solution. Let P(n) be the statement $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Now P(1) means $1^3 = \left(\frac{1(1+1)}{2}\right)^2$ *i.e.* $1 = 1^2$ or $1 = 1$, which is true

$\Rightarrow$ P(1) is true.

Let P(m) be true

i.e. $1^3 + 2^3 + 3^3 + \ldots + m^3 = \left(\frac{m(m+1)}{2}\right)^2$ ...(i)

For P(m + 1) : $1^3 + 2^3 + 3^3 + \ldots + (m+1)^3$

$$= (1^3 + 2^3 + 3^3 + \ldots + m^3) + (m+1)^3$$

$$= \left(\frac{m(m+1)}{2}\right)^2 + (m+1)^3 \qquad \text{(using } (i))$$

$$= (m+1)^2 \left[\frac{m^2}{4} + (m+1)\right]$$

$$= (m+1)^2 \frac{m^2 + 4m + 4}{4} = \left(\frac{(m+1)(m+2)}{2}\right)^2$$

$\Rightarrow$ P(m + 1) is true.

Hence, by induction, P(n) is true for all $n \in \mathbf{N}$.

Example 6. *For all $n \in \mathbf{N}$, using principle of mathematical induction prove that :*

$$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+n} = \frac{2n}{n+1}. \qquad \text{(NCERT)}$$

Solution. Let P(n) be the statement

$$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+n} = \frac{2n}{n+1}.$$

Now P(1) means $1 = \frac{2.1}{1+1}$ *i.e.* $1 = \frac{2}{2}$ or 1 = 1, which is true

$\Rightarrow$ P(1) is true.

Let P(m) be true

i.e. $$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+m} = \frac{2m}{m+1} \qquad \ldots(i)$$

For P(m + 1) : $$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+m} + \frac{1}{1+2+3+\ldots+(m+1)}$$

$$= \left(1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots \frac{1}{1+2+3+\ldots+m}\right) + \frac{1}{1+2+3+\ldots+(m+1)}$$

$$= \frac{2m}{m+1} + \frac{1}{1+2+3+\ldots+(m+1)} \qquad \text{(using } (i))$$

$$= \frac{2m}{m+1} + \frac{1}{\frac{(m+1)(m+2)}{2}}$$

$\left[\because 1 + 2 + 3 + \ldots + (m + 1) = \text{sum of } (m + 1) \text{ terms of an A.P. whose 1st term is 1 and common difference is 1}\right.$

$$\left. = \frac{m+1}{2}\,(2.1 + (\overline{m+1} - 1).1) = \frac{(m+1)(m+2)}{2}\right]$$

$$= \frac{2m}{m+1} + \frac{2}{(m+1)(m+2)} = \frac{2}{m+1}\left(m + \frac{1}{m+2}\right)$$

$$= \frac{2}{m+1}.\frac{m^2+2m+1}{m+2} = \frac{2(m+1)^2}{(m+1)(m+2)} = \frac{2(m+1)}{m+2}$$

$\Rightarrow$ P(m + 1) is true.

Hence, by principle of mathematical induction, P(n) is true for all $n \in \mathbf{N}$.

Example 7. *Using principle of mathematical induction prove that*

$$1.3 + 2.3^2 + 3.3^3 + \ldots + n.3^n = \frac{(2n-1)3^{n+1}+3}{4}, \text{ for all } n \in \mathbf{N}. \qquad \text{(NCERT)}$$

Solution. Let P(n) be the statement

$$1.3 + 2.3^2 + 3.3^3 + \ldots + n.3^n = \frac{(2n-1)3^{n+1}+3}{4}.$$

Now P(1) means $1.3 = \frac{(2.1-1)3^{1+1}+3}{4}$ *i.e.* $3 = \frac{1.9+3}{4}$ *i.e.* $3 = \frac{12}{4}$ *i.e.* 3 = 3, which is true

$\Rightarrow$ P(1).

Let P(m) be true

i.e. $$1.3 + 2.3^2 + 3.3^3 + \ldots + m.3^m = \frac{(2m-1)3^{m+1}+3}{4} \qquad \ldots(i)$$

For P(m + 1) : $1.3 + 2.3^2 + 3.3^3 + \ldots + (m+1).3^{m+1}$

$$= (1.3 + 2.3^2 + 3.3^3 + \ldots + m.3^m) + (m+1).\,3^{m+1}$$

$$= \frac{(2m-1).3^{m+1}+3}{4} + (m+1).3^{m+1} \qquad \text{(using } (i)\text{)}$$

$$= \frac{(2m-1).3^{m+1}+3+(4m+4).3^{m+1}}{4}$$

$$= \frac{(6m+3).3^{m+1}+3}{4} = \frac{(2m+1).3.3^{m+1}+3}{4}$$

$$= \frac{(2(m+1)-1).3^{m+2}+3}{4}$$

$\Rightarrow$ P $(m + 1)$ is true.

Hence, by principle of mathematical induction, P (n) is true for all $n \in$ **N**.

Example 8. *Use induction to prove that*

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2n+1}{n^2}\right) = (n+1)^2, \textit{ for all } n \in \mathbf{N}. \qquad \textit{(NCERT)}$$

Solution. Let P(n) be the statement

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2n+1}{n^2}\right) = (n+1)^2.$$

Now P(1) means $1 + \frac{3}{1} = (1+1)^2$ *i.e.* $4 = 4$, which is true

$\Rightarrow$ P(1) is true.

Let P(m) be true

i.e. $$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2m+1}{m^2}\right) = (m+1)^2 \qquad \ldots(i)$$

For P$(m + 1)$: $$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2(m+1)+1}{(m+1)^2}\right)$$

$$= \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2m+1}{m^2}\right)\left(1+\frac{2m+3}{(m+1)^2}\right)$$

$$= (m+1)^2\left(1+\frac{2m+3}{(m+1)^2}\right) \qquad \text{(using } (i)\text{)}$$

$$= (m+1)^2 . \frac{(m+1)^2+2m+3}{(m+1)^2} = (m+1)^2 + 2m + 3$$

$$= m^2 + 4m + 4 = (m+2)^2 = (\overline{m+1} + 1)^2$$

$\Rightarrow$ P $(m + 1)$ is true.

Hence, by principle of mathematical induction, P (n) is true for all $n \in$ **N**.

Example 9. *Using induction, prove that :*

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n} \textit{ for all natural numbers } n,\ n \geq 2.$$

(*NCERT Examplar Problems*)

Solution. Let P(n) be the statement

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n},\ n \geq 2.$$

Now P(2) means $\left(1-\frac{1}{2^2}\right) = \frac{2+1}{2\times 2}$ *i.e.* $1-\frac{1}{4} = \frac{3}{4}$ *i.e.* $\frac{3}{4} = \frac{3}{4}$, which is true

$\Rightarrow$ P(2) is true.

Let P(m) be true

i.e. $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{m^2}\right)=\frac{m+1}{2m}$ …(i)

For P(m + 1) : $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{m^2}\right)\left(1-\frac{1}{(m+1)^2}\right)$

$$= \frac{m+1}{2m}\times\frac{(m+1)^2-1}{(m+1)^2} \quad \text{(using } (i))$$

$$= \frac{m^2+2m}{2m(m+1)} = \frac{m+2}{2(m+1)} = \frac{(m+1)+1}{2(m+1)}$$

⇒ P (m + 1) is true.

Hence, by induction, P(n) is true for natural numbers n, $n \geq 2$.

NOTE

In the above example, the given statement starts with n = 2.

Example 10. *By principle of mathematical induction, prove that for all $n \in$ **N***

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{6n+4}. \quad \textit{(NCERT)}$$

Solution. Let P(n) be the statement

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{6n+4}.$$

Now P(1) means $\frac{1}{2.5}=\frac{1}{6.1+4}$ *i.e.* $\frac{1}{10}=\frac{1}{10}$, which is true

⇒ P(1) is true.

Let P(m) be true

i.e. $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3m-1)(3m+2)}=\frac{m}{6m+4}$ …(i)

For P(m + 1) : $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3(m+1)-1)(3(m+1)+2)}$

$$= \left[\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3m-1)(3m+2)}\right]+\frac{1}{(3m+2)(3m+5)}$$

$$= \frac{m}{6m+4}+\frac{1}{(3m+2)(3m+5)} \quad \text{(using } (i))$$

$$= \frac{m}{2(3m+2)}+\frac{1}{(3m+2)(3m+5)}$$

$$= \frac{m(3m+5)+2}{2(3m+2)(3m+5)} = \frac{3m^2+5m+2}{2(3m+2)(3m+5)}$$

$$= \frac{(3m+2)(m+1)}{2(3m+2)(3m+5)} = \frac{m+1}{2(3m+5)} = \frac{m+1}{6(m+1)+4}$$

⇒ P(m + 1) is true.

Hence, by principle of mathematical induction, P(n) is true for all $n \in$ **N**.

EXERCISE 4.2

Using the principle of mathematical induction prove that (1 to 26) for all $n \in \mathbf{N}$:

1. $2 + 4 + 6 + \ldots + 2n = n^2 + n.$ (*NCERT Examplar Problems*)

2. $1 + 4 + 7 + \ldots + (3n - 2) = \frac{1}{2} n (3n - 1).$

3. $3x + 6x + 9x + \ldots$ to n terms $= \frac{3}{2} n (n + 1)x.$

4. $1^2 + 3^2 + 5^2 + \ldots$ to n terms $= \frac{n(4n^2 - 1)}{3}.$ (*NCERT*)

5. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}.$ (*NCERT*)

6. $3.6 + 6.9 + 9.12 + \ldots + 3n(3n + 3) = 3n(n + 1)(n + 2).$

7. $1.2 + 2.2^2 + 3.2^3 + \ldots + n.2^n = (n - 1).2^{n+1} + 2.$ (*NCERT*)

8. $1.2.3 + 2.3.4 + 3.4.5 + \ldots + n(n + 1)(n + 2) = \frac{n(n + 1)(n + 2)(n + 3)}{4}.$ (*NCERT*)

9. $\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) \ldots \left(1 + \frac{1}{n}\right) = n + 1$ (*NCERT*)

10. $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) \cdots \left(1 - \frac{1}{n+1}\right) = \frac{1}{n+1}.$

11. $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots$ to n terms $= \frac{n}{n+1}$

12. $\frac{1}{3.6} + \frac{1}{6.9} + \frac{1}{9.12} + \ldots + \frac{1}{3n(3n+3)} = \frac{n}{9(n+1)}.$

13. $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots$ to n terms $= \frac{n}{2n+1}.$

14. $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \ldots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}.$ (*NCERT*)

15. $\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \ldots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}.$ (*NCERT*)

16. $2^{2n} - 1$ is divisible by 3. (*NCERT Examplar Problems*)

17. $2^{3n} - 1$ is divisible by 7. (*NCERT Examplar Problems*)

18. 3^{2n} when divided by 8, leaves the remainder 1.

19. $10^{2n-1} + 1$ is divisible by 11. (*NCERT*)

20. $7^n - 3^n$ is divisible by 4. (*NCERT*)

21. $4^n + 15n - 1$ is divisible by 9.

22. $3^{2n+2} - 8n - 9$ is a multiple of 64. (*NCERT*)

23. $n(n + 1)(n + 5)$ is a multiple of 3. (*NCERT*)

24. $n^3 + (n + 1)^3 + (n + 2)^3$ is a multiple of 9.

25. $x^{2n-1} - 1$ is divisible by $(x - 1)$, $x \neq 1$.

26. $3^n > n$ for all $n \in \mathbf{N}$.

27. Prove, by induction, that $5^{n+1} + 4.6^n$ when divided by 20 leaves the remainder 9 for all $n \in \mathbf{N}$.

28. Prove, by induction, that $8.7^n + 4^{n+2}$ is divisible by 24 but not by 48 for all $n \in \mathbf{N}$.