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## COMPLEX NUMBERS AND QUADRATIC EQUATIONS

### INTRODUCTION

We know that  $x^2 \geq 0$  for all  $x \in \mathbf{R}$  i.e. the square of a real number (whether positive, negative or zero) is non-negative. Hence the equations  $x^2 = -1$ ,  $x^2 = -5$ ,  $x^2 + 7 = 0$  etc. are not solvable in *real number system*. Thus, there is a need to extend the real number system to a larger system so that we can have solutions of such equations. In fact, our main objective is to solve the quadratic equation  $ax^2 + bx + c = 0$ , where  $a, b, c \in \mathbf{R}$  and the discriminant  $= b^2 - 4ac < 0$ , which is not possible in real number system. In this chapter, we shall extend the real number system to a larger system called **complex number system** so that the solutions of quadratic equations  $ax^2 + bx + c = 0$ , where  $a, b, c$  are real numbers are possible. We shall also solve quadratic equations with complex coefficients.

### 5.1 COMPLEX NUMBERS

We know that the equation  $x^2 + 1 = 0$  is not solvable in the real number system i.e. it has no real roots. Many mathematicians indicated the square roots of negative numbers, but **Euler** was the first to introduce the symbol  $i$  (read 'iota') to represent  $\sqrt{-1}$ , and he defined  $i^2 = -1$ .

It follows that  $i$  is a solution of the equation  $x^2 + 1 = 0$ . Also  $(-i)^2 = i^2 = -1$ . Thus the equation  $x^2 + 1 = 0$  has two solutions,  $x = \pm i$ , where  $i = \sqrt{-1}$ .

The number  $i$  is called an **imaginary number**. In general, the square roots of all negative real numbers are called **imaginary numbers**. Thus  $\sqrt{-1}$ ,  $\sqrt{-5}$ ,  $\sqrt{-\frac{9}{4}}$  etc. are all imaginary numbers.

#### Complex number

A number of the form  $a + ib$ , where  $a$  and  $b$  are real numbers, is called a **complex number**.

For example,  $3 + 5i$ ,  $-2 + 3i$ ,  $-2 + i\sqrt{5}$ ,  $7 + i\left(-\frac{2}{3}\right)$  are all complex numbers.

The system of numbers  $\mathbf{C} = \{z; z = a + ib; a, b \in \mathbf{R}\}$  is called the set of **complex numbers**.

#### Standard form of a complex number

If a complex number is expressed in the form  $a + ib$  where  $a, b \in \mathbf{R}$  and  $i = \sqrt{-1}$ , then it is said to be in the **standard form**.

For example, the complex numbers  $2 + 5i$ ,  $-3 + \sqrt{2}i$ ,  $-\frac{2}{3} - 7i$  are all in the standard form.

#### Real and imaginary parts of a complex number

If  $z = a + ib$  ( $a, b \in \mathbf{R}$ ) is a complex number, then  $a$  is called the **real part**, denoted by  $\text{Re}(z)$  and  $b$  is called **imaginary part**, denoted by  $\text{Im}(z)$ .

For example :

- (i) If  $z = 2 + 3i$ , then  $\operatorname{Re}(z) = 2$  and  $\operatorname{Im}(z) = 3$ .
- (ii) If  $z = -3 + \sqrt{5}i$ , then  $\operatorname{Re}(z) = -3$  and  $\operatorname{Im}(z) = \sqrt{5}$ .
- (iii) If  $z = 7$ , then  $z = 7 + 0i$ , so that  $\operatorname{Re}(z) = 7$  and  $\operatorname{Im}(z) = 0$ .
- (iv) If  $z = -5i$ , then  $z = 0 + (-5)i$ , so that  $\operatorname{Re}(z) = 0$  and  $\operatorname{Im}(z) = -5$ .

Note that imaginary part of a complex number is a real number.

In  $z = a + ib$  ( $a, b \in \mathbf{R}$ ), if  $b = 0$  then  $z = a$ , which is a **real number**. If  $a = 0$  and  $b \neq 0$ , then  $z = ib$ , which is called **purely imaginary number**. If  $b \neq 0$ , then  $z = a + ib$  is **non-real complex number**. Since every real number  $a$  can be written as  $a + 0i$ , we see that  $\mathbf{R} \subset \mathbf{C}$  i.e. the set of real numbers  $\mathbf{R}$  is a **proper subset** of  $\mathbf{C}$ , the set of complex numbers.

Note that  $\sqrt{3}$ ,  $0$ ,  $2$ ,  $\pi$  are real numbers;  $3 + 2i$ ,  $3 - 2i$  etc. are non-real complex numbers;  $2i$ ,  $-\sqrt{2}i$  etc. are purely imaginary numbers.

### Equality of two complex numbers

Two complex numbers  $z_1 = a + ib$  and  $z_2 = c + id$  are called **equal**, written as  $z_1 = z_2$ , if and only if  $a = c$  and  $b = d$ .

For example, if the complex numbers  $z_1 = a + ib$  and  $z_2 = -3 + 5i$  are equal, then  $a = -3$  and  $b = 5$ .

### 5.1.1 Algebra of complex numbers

In this section, we shall define the usual mathematical operations — addition, subtraction, multiplication, division, square, power etc. on complex numbers and will develop the algebra of complex numbers.

#### Addition of two complex numbers

Let  $z_1 = a + ib$  and  $z_2 = c + id$  be any two complex numbers, then their sum  $z_1 + z_2$  is defined as  $z_1 + z_2 = (a + c) + i(b + d)$ .

For example, let  $z_1 = 2 + 3i$  and  $z_2 = -5 + 4i$ , then

$$z_1 + z_2 = (2 + (-5)) + (3 + 4)i = -3 + 7i.$$

#### Properties of addition of complex numbers

##### (i) Closure property

The sum of two complex numbers is a complex number i.e. if  $z_1$  and  $z_2$  are any two complex numbers, then  $z_1 + z_2$  is always a complex number.

##### (ii) Addition of complex numbers is commutative

If  $z_1$  and  $z_2$  are any two complex numbers, then  $z_1 + z_2 = z_2 + z_1$ .

##### (iii) Addition of complex numbers is associative

If  $z_1$ ,  $z_2$  and  $z_3$  are any three complex numbers, then

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3).$$

##### (iv) The existence of additive identity

Let  $z = x + iy$ ,  $x, y \in \mathbf{R}$ , be any complex number, then

$$(x + iy) + (0 + i0) = (x + 0) + i(y + 0) = x + iy \text{ and}$$

$$(0 + i0) + (x + iy) = (0 + x) + i(0 + y) = x + iy$$

$$\Rightarrow (x + iy) + (0 + i0) = x + iy = (0 + i0) + (x + iy).$$

Therefore,  $0 + i0$  acts as the additive identity. It is simply written as  $0$ .

Thus,  $z + 0 = z = 0 + z$  for all complex numbers  $z$ .

**(v) The existence of additive inverse**

For a complex number  $z = a + ib$ , its negative is defined as

$$-z = (-a) + i(-b) = -a - ib.$$

Note that  $z + (-z) = (a - a) + i(b - b) = 0 + i0 = 0$ .

Thus  $-z$  acts as additive inverse of  $z$ .

**Subtraction of complex numbers**

Let  $z_1 = a + ib$  and  $z_2 = c + id$  be any two complex numbers, then the subtraction of  $z_2$  from  $z_1$  is defined as

$$\begin{aligned} z_1 - z_2 &= z_1 + (-z_2) \\ &= (a + ib) + (-c - id) \\ &= (a - c) + i(b - d). \end{aligned}$$

For example, let  $z_1 = 2 + 3i$  and  $z_2 = -1 + 4i$ , then

$$\begin{aligned} z_1 - z_2 &= (2 + 3i) - (-1 + 4i) \\ &= (2 + 3i) + (1 - 4i) \\ &= (2 + 1) + (3 - 4)i = 3 - i. \end{aligned}$$

$$\begin{aligned} \text{and } z_2 - z_1 &= (-1 + 4i) - (2 + 3i) \\ &= (-1 + 4i) + (-2 - 3i) \\ &= (-1 - 2) + (4 - 3)i = -3 + i. \end{aligned}$$

**Multiplication of two complex numbers**

Let  $z_1 = a + ib$  and  $z_2 = c + id$  be any two complex numbers, then their product  $z_1 z_2$  is defined as

$$z_1 z_2 = (ac - bd) + i(ad + bc).$$

Note that intuitively,

$$(a + ib)(c + id) = ac + ibc + iad + i^2 bd ; \text{ now put } i^2 = -1, \text{ thus}$$

$$(a + ib)(c + id) = ac + i(bc + ad) - bd = (ac - bd) + i(ad + bc).$$

For example, let  $z_1 = 3 + 7i$  and  $z_2 = -2 + 5i$ , then

$$\begin{aligned} z_1 z_2 &= (3 + 7i)(-2 + 5i) \\ &= (3 \times (-2) - 7 \times 5) + i(3 \times 5 + 7 \times (-2)) \\ &= -41 + i. \end{aligned}$$

**Properties of multiplication of complex numbers****(i) Closure property**

The product of two complex numbers is a complex number *i.e.* if  $z_1$  and  $z_2$  are any two complex numbers, then  $z_1 z_2$  is always a complex number.

**(ii) Multiplication of complex numbers is commutative**

If  $z_1$  and  $z_2$  are any two complex numbers, then  $z_1 z_2 = z_2 z_1$ .

**(iii) Multiplication of complex numbers is associative**

If  $z_1, z_2$  and  $z_3$  are any three complex numbers, then  $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ .

**(iv) The existence of multiplicative identity**

Let  $z = x + iy$ ,  $x, y \in \mathbf{R}$ , be any complex number, then

$$(x + iy)(1 + i0) = (x.1 - y.0) + i(x.0 + y.1) = x + iy \text{ and}$$

$$(1 + i0)(x + iy) = (1.x - 0.y) + i(1.y + 0.x) = x + iy$$

$$\Rightarrow (x + iy)(1 + i0) = x + iy = (1 + i0)(x + iy).$$

Therefore,  $1 + i0$  acts as the multiplicative identity. It is simply written as 1.

Thus  $z.1 = z = 1.z$  for all complex numbers  $z$ .

**(v) Existence of multiplicative inverse**

For every non-zero complex number  $z = a + ib$ , we have the complex number

$\frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$  (denoted by  $z^{-1}$  or  $\frac{1}{z}$ ) such that

$$z \cdot \frac{1}{z} = 1 = \frac{1}{z} \cdot z$$

(check it)

$\frac{1}{z}$  is called the multiplicative inverse of  $z$ .

Note that intuitively,  $\frac{1}{a+ib} = \frac{1}{a+ib} \times \frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$ .

(vi) **Multiplication of complex numbers is distributive over addition of complex numbers**

If  $z_1, z_2$  and  $z_3$  are any three complex numbers, then

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3$$

$$\text{and} \quad (z_1 + z_2)z_3 = z_1z_3 + z_2z_3.$$

These results are known as **distributive laws**.

### Division of complex numbers

Division of a complex number  $z_1 = a + ib$  by  $z_2 = c + id \neq 0$  is defined as

$$\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2} = z_1 \cdot z_2^{-1} = (a + ib) \cdot \left( \frac{c}{c^2 + d^2} - i\frac{d}{c^2 + d^2} \right) = \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}.$$

Note that intuitively,

$$\frac{z_1}{z_2} = \frac{a + ib}{c + id} = \frac{a + ib}{c + id} \times \frac{c - id}{c - id} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}.$$

For example, if  $z_1 = 3 + 4i$  and  $z_2 = 5 - 6i$ , then

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{3+4i}{5-6i} = \frac{3+4i}{5-6i} \times \frac{5+6i}{5+6i} = \frac{(3 \times 5 - 4 \times 6) + (3 \times 6 + 4 \times 5)i}{5^2 - 6^2 \times i^2} \\ &= \frac{-9 + 38i}{25 + 36} = -\frac{9}{61} + \frac{38}{61}i. \end{aligned}$$

### Integral powers of a complex number

If  $z$  is any complex number, then positive integral powers of  $z$  are defined as

$$z^1 = z, z^2 = z.z, z^3 = z^2.z, z^4 = z^3.z \text{ and so on.}$$

If  $z$  is any non-zero complex number, then negative integral powers of  $z$  are defined as :

$$z^{-1} = \frac{1}{z}, z^{-2} = \frac{1}{z^2}, z^{-3} = \frac{1}{z^3} \text{ etc.}$$

If  $z \neq 0$ , then  $z^0 = 1$ .

#### 5.1.2 Powers of $i$

Integral power of  $i$  are defined as :

$$i^0 = 1, i^1 = i, i^2 = -1,$$

$$i^3 = i^2.i = (-1).i = -i,$$

$$i^4 = (i^2)^2 = (-1)^2 = 1,$$

$$i^5 = i^4.i = 1.i = i,$$

$$i^6 = i^4.i^2 = 1.(-1) = -1, \text{ and so on.}$$

$$i^{-1} = \frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{-1} = -i$$

**Remember that**  $\frac{1}{i} = -i$

$$i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1,$$

$$i^{-3} = \frac{1}{i^3} = \frac{1}{i^3} \times \frac{i}{i} = \frac{i}{i^4} = \frac{i}{1} = i$$

$$i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1, \text{ and so on.}$$

**Note that**  $i^4 = 1$  and  $i^{-4} = 1$ . It follows that for any integer  $k$ ,

$$i^{4k} = 1, i^{4k+1} = i, i^{4k+2} = i^2 = -1, i^{4k+3} = i^3 = -i.$$

Also, we note that  $i^2 = -1$  and  $(-i)^2 = i^2 = -1$ .

Therefore,  $i$  and  $-i$  are both square roots of  $-1$ . However, by the symbol  $\sqrt{-1}$ , we shall mean  $i$  only i.e.  $\sqrt{-1} = i$ .

We observe that  $i$  and  $-i$  are both the solutions of the equation  $x^2 + 1 = 0$ .

$$\text{Similarly, } (\sqrt{5}i)^2 = (\sqrt{5})^2 i^2 = 5(-1) = -5,$$

$$\text{and } (-\sqrt{5}i)^2 = (-\sqrt{5})^2 i^2 = 5(-1) = -5.$$

Therefore,  $\sqrt{5}i$  and  $-\sqrt{5}i$  are both square roots of  $-5$ . However, by the symbol  $\sqrt{-5}$ , we shall mean  $\sqrt{5}i$  only i.e.  $\sqrt{-5} = \sqrt{5}i$ .

**In general, if  $a$  is any positive real number, then  $\sqrt{-a} = \sqrt{a}i$ .**

We already know that  $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$  for all positive real numbers  $a$  and  $b$ . This result is also true when either  $a > 0, b < 0$  or  $a < 0, b > 0$ . But what if  $a < 0, b < 0$ ? Let us examine :

we note that  $i^2 = i \times i = \sqrt{-1} \sqrt{-1} = \sqrt{(-1)(-1)}$  (by assuming  $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$  for all real numbers)  $= \sqrt{1} = 1$ . Thus, we get  $i^2 = 1$  which is contrary to the fact that  $i^2 = -1$ .

Therefore,  $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$  is **not true** when  $a$  and  $b$  are both negative real numbers.

Further, if any of  $a$  and  $b$  is zero, then  $\sqrt{a} \times \sqrt{b} = \sqrt{ab} = 0$ .

### 5.1.3 Identities

If  $z_1$  and  $z_2$  are any two complex numbers, then the following results hold :

$$(i) (z_1 + z_2)^2 = z_1^2 + 2z_1z_2 + z_2^2$$

$$(ii) (z_1 - z_2)^2 = z_1^2 - 2z_1z_2 + z_2^2$$

$$(iii) (z_1 + z_2)(z_1 - z_2) = z_1^2 - z_2^2$$

$$(iv) (z_1 + z_2)^3 = z_1^3 + 3z_1^2z_2 + 3z_1z_2^2 + z_2^3$$

$$(v) (z_1 - z_2)^3 = z_1^3 - 3z_1^2z_2 + 3z_1z_2^2 - z_2^3.$$

**Proof.** (i)  $(z_1 + z_2)^2 = (z_1 + z_2)(z_1 + z_2)$

$$= (z_1 + z_2)z_1 + (z_1 + z_2)z_2 \quad \text{(Distributive law)}$$

$$= z_1^2 + z_2z_1 + z_1z_2 + z_2^2 \quad \text{(Distributive law)}$$

$$= z_1^2 + z_1z_2 + z_1z_2 + z_2^2 \quad \text{(Commutative law)}$$

$$= z_1^2 + 2z_1z_2 + z_2^2.$$

We leave the proofs of the other results for the reader.

### 5.1.4 Modulus of a complex number

Modulus of a complex number  $z = a + ib$ , denoted by  $\text{mod}(z)$  or  $|z|$ , is defined as

$$|z| = \sqrt{a^2 + b^2}, \text{ where } a = \text{Re}(z), b = \text{Im}(z).$$

Sometimes,  $|z|$  is called **absolute value** of  $z$ . Note that  $|z| \geq 0$ .

For example :

$$(i) \text{ If } z = -3 + 5i, \text{ then } |z| = \sqrt{(-3)^2 + 5^2} = \sqrt{34}.$$

$$(ii) \text{ If } z = 3 - \sqrt{7}i, \text{ then } |z| = \sqrt{3^2 + (-\sqrt{7})^2} = \sqrt{9 + 7} = 4.$$

#### Properties of modulus of a complex number

If  $z, z_1$  and  $z_2$  are complex numbers, then

$$(i) |-z| = |z| \quad (ii) |z| = 0 \text{ if and only if } z = 0$$

$$(iii) |z_1 z_2| = |z_1| |z_2| \quad (iv) \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, \text{ provided } z_2 \neq 0.$$

**Proof.** (i) Let  $z = a + ib$ , where  $a, b \in \mathbf{R}$ , then  $-z = -a - ib$ .

$$\therefore |-z| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|.$$

$$(ii) \text{ Let } z = a + ib, \text{ then } |z| = \sqrt{a^2 + b^2}.$$

$$\text{Now } |z| = 0 \text{ iff } \sqrt{a^2 + b^2} = 0$$

$$\text{i.e. iff } a^2 + b^2 = 0 \text{ i.e. iff } a^2 = 0 \text{ and } b^2 = 0$$

$$\text{i.e. iff } a = 0 \text{ and } b = 0 \text{ i.e. iff } z = 0 + i0$$

$$\text{i.e. iff } z = 0.$$

$$(iii) \text{ Let } z_1 = a + ib, \text{ and } z_2 = c + id, \text{ then}$$

$$z_1 z_2 = (ac - bd) + i(ad + bc).$$

$$\begin{aligned} \therefore |z_1 z_2| &= \sqrt{(ac - bd)^2 + (ad + bc)^2} \\ &= \sqrt{a^2 c^2 + b^2 d^2 - 2abcd + a^2 d^2 + b^2 c^2 + 2abcd} \\ &= \sqrt{(a^2 + b^2)(c^2 + d^2)} \\ &= \sqrt{a^2 + b^2} \sqrt{c^2 + d^2} \quad (\because a^2 + b^2 \geq 0, c^2 + d^2 \geq 0) \\ &= |z_1| |z_2|. \end{aligned}$$

$$(iv) \text{ Here } z_2 \neq 0 \Rightarrow |z_2| \neq 0.$$

$$\text{Let } \frac{z_1}{z_2} = z_3 \Rightarrow z_1 = z_2 z_3 \Rightarrow |z_1| = |z_2 z_3|$$

$$\Rightarrow |z_1| = |z_2| |z_3| \quad (\text{using part (iii)})$$

$$\Rightarrow \frac{|z_1|}{|z_2|} = |z_3| \Rightarrow \frac{|z_1|}{|z_2|} = \left| \frac{z_1}{z_2} \right| \quad \left( \because z_3 = \frac{z_1}{z_2} \right)$$

#### REMARK

From (iii), on replacing both  $z_1$  and  $z_2$  by  $z$ , we get

$$|z z| = |z| |z| \text{ i.e. } |z^2| = |z|^2.$$

$$\text{Similary, } |z^3| = |z^2 z| = |z^2| |z| = |z|^2 |z| = |z|^3 \text{ etc.}$$

### 5.1.5 Conjugate of a complex number

Conjugate of a complex number  $z = a + ib$ , denoted by  $\bar{z}$ , is defined as

$$\bar{z} = a - ib \text{ i.e. } \overline{a + ib} = a - ib.$$

For example :

$$(i) \quad \overline{2 + 5i} = 2 - 5i, \quad \overline{2 - 5i} = 2 + 5i$$

$$(ii) \quad \overline{-3 - 7i} = -3 + 7i, \quad \overline{-3 + 7i} = -3 - 7i.$$

#### Properties of conjugate of a complex number

If  $z, z_1$  and  $z_2$  are complex numbers, then

$$(i) \quad \overline{(\bar{z})} = z$$

$$(ii) \quad \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

$$(iii) \quad \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$$

$$(iv) \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

$$(v) \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, \text{ provided } z_2 \neq 0$$

$$(vi) \quad |\bar{z}| = |z|$$

$$(vii) \quad z \bar{z} = |z|^2$$

$$(viii) \quad z^{-1} = \frac{\bar{z}}{|z|^2}, \text{ provided } z \neq 0.$$

**Proof.** (i) Let  $z = a + ib$ , where  $a, b \in \mathbf{R}$ , so that  $\bar{z} = a - ib$ .

$$\therefore \overline{(\bar{z})} = \overline{a - ib} = a + ib = z.$$

(ii) Let  $z_1 = a + ib$  and  $z_2 = c + id$ , then

$$\begin{aligned} \overline{z_1 + z_2} &= \overline{(a + ib) + (c + id)} = \overline{(a + c) + i(b + d)} \\ &= (a + c) - i(b + d) = (a - ib) + (c - id) = \bar{z}_1 + \bar{z}_2. \end{aligned}$$

(iii) Let  $z_1 = a + ib$  and  $z_2 = c + id$ , then

$$\begin{aligned} \overline{z_1 - z_2} &= \overline{(a + ib) - (c + id)} = \overline{(a - c) + i(b - d)} \\ &= (a - c) - i(b - d) = (a - ib) - (c - id) \\ &= \bar{z}_1 - \bar{z}_2. \end{aligned}$$

(iv) Let  $z_1 = a + ib$  and  $z_2 = c + id$ , then

$$\begin{aligned} \overline{z_1 z_2} &= \overline{(a + ib)(c + id)} = \overline{(ac - bd) + i(ad + bc)} \\ &= (ac - bd) - i(ad + bc). \end{aligned}$$

$$\text{Also } \overline{\bar{z}_1 \bar{z}_2} = \overline{(a - ib)(c - id)} = \overline{(ac - bd) - i(ad + bc)}.$$

$$\text{Hence } \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2.$$

(v) Here  $z_2 \neq 0 \Rightarrow \bar{z}_2 \neq 0$ .

$$\text{Let } \frac{z_1}{z_2} = z_3 \Rightarrow z_1 = z_2 z_3 \Rightarrow \bar{z}_1 = \overline{z_2 z_3}$$

$$\Rightarrow \bar{z}_1 = \bar{z}_2 \bar{z}_3$$

(using part (iv))

$$\Rightarrow \frac{\bar{z}_1}{\bar{z}_2} = \bar{z}_3 \Rightarrow \frac{\bar{z}_1}{\bar{z}_2} = \overline{\left(\frac{z_1}{z_2}\right)}$$

$$\left( \because z_3 = \frac{z_1}{z_2} \right)$$

(vi) Let  $z = a + ib$ , then  $\bar{z} = a - ib$ .

$$\therefore |\bar{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|.$$

(vii) Let  $z = a + ib$ , then  $\bar{z} = a - ib$ .

$$\begin{aligned}\therefore z \bar{z} &= (a + ib)(a - ib) \\ &= (aa - b(-b)) + i(a(-b) + ba) \quad (\text{Def. of multiplication}) \\ &= (a^2 + b^2) + i \cdot 0 \\ &= a^2 + b^2 = \left(\sqrt{a^2 + b^2}\right)^2 = |z|^2.\end{aligned}$$

**Remember that**  $(a + ib)(a - ib) = a^2 + b^2$ .

(viii) Let  $z = a + ib \neq 0$ , then  $|z| \neq 0$ .

$$\begin{aligned}\therefore z \bar{z} &= (a + ib)(a - ib) = a^2 + b^2 = |z|^2 \\ \Rightarrow \frac{z \bar{z}}{|z|^2} &= 1 \Rightarrow \frac{\bar{z}}{|z|^2} = \frac{1}{z} = z^{-1}\end{aligned}$$

$$\text{Thus, } z^{-1} = \frac{\bar{z}}{|z|^2}, \text{ provided } z \neq 0.$$

### REMARK

From (iv), on replacing both  $z_1$  and  $z_2$  by  $z$ , we get

$$\overline{z z} = \bar{z} \bar{z} \text{ i.e. } \overline{z^2} = (\bar{z})^2.$$

Similarly,  $\overline{(z^3)} = \overline{(z^2 z)} = (\bar{z}^2) \bar{z} = (\bar{z})^2 \bar{z} = (\bar{z})^3$  etc.

### NOTE

The order relations 'greater than' and 'less than' are not defined for complex numbers i.e. the inequalities  $2 + 3i > -2 + 5i$ ,  $4i \geq 1 - 2i$ ,  $-1 + 3i < 5$  etc. are meaningless.

## ILLUSTRATIVE EXAMPLES

**Example 1.** A student says

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1} \sqrt{-1} = i \cdot i = i^2 = -1. \text{ Thus } 1 = -1.$$

Where is the fault?

**Solution.**  $1 = \sqrt{1} = \sqrt{(-1)(-1)}$  is true, but  $\sqrt{(-1)(-1)} = \sqrt{-1} \sqrt{-1}$  is wrong.

Because if both  $a, b$  are negative real numbers, then  $\sqrt{a} \sqrt{b} = \sqrt{ab}$  is not true.

**Example 2.** If  $z = \sqrt{37} + \sqrt{-19}$ , find  $\text{Re}(z)$ ,  $\text{Im}(z)$ ,  $\bar{z}$  and  $|z|$ .

**Solution.** Given  $z = \sqrt{37} + \sqrt{-19} = \sqrt{37} + i\sqrt{19}$ .

$$\therefore \text{Re}(z) = \sqrt{37} \text{ and } \text{Im}(z) = \sqrt{19}.$$

$$\bar{z} = \overline{\sqrt{37} + i\sqrt{19}} = \sqrt{37} - i\sqrt{19}.$$

$$|z| = \sqrt{(\sqrt{37})^2 + (\sqrt{19})^2} = \sqrt{37 + 19} = \sqrt{56} = 2\sqrt{14}.$$

**Example 3.** If  $4x + i(3x - y) = 3 - 6i$  and  $x, y$  are real numbers, then find the values of  $x$  and  $y$ . (NCERT)

**Solution.** Given  $4x + i(3x - y) = 3 - 6i$

$$\Rightarrow 4x + i(3x - y) = 3 + i(-6).$$

Equating real and imaginary parts on both sides, we get

$$4x = 3 \text{ and } 3x - y = -6$$

$$\Rightarrow x = \frac{3}{4} \text{ and } 3 \times \frac{3}{4} - y = -6$$

$$\Rightarrow x = \frac{3}{4} \text{ and } y = 6 + \frac{9}{4} = \frac{33}{4}.$$

$$\text{Hence } x = \frac{3}{4} \text{ and } y = \frac{33}{4}.$$

**Example 4.** For what real values of  $x$  and  $y$  are the following numbers equal

$$(i) (1 + i) y^2 + (6 + i) \text{ and } (2 + i) x$$

$$(ii) x^2 - 7x + 9yi \text{ and } y^2i + 20i - 12 ?$$

**Solution.** (i) Given  $(1 + i) y^2 + (6 + i) = (2 + i) x$

$$\Rightarrow (y^2 + 6) + i(y^2 + 1) = 2x + ix$$

$$\Rightarrow y^2 + 6 = 2x \text{ and } y^2 + 1 = x$$

$$\Rightarrow x = 5 \text{ and } y^2 = 4 \Rightarrow x = 5 \text{ and } y = \pm 2.$$

Hence, the required values of  $x$  and  $y$  are

$$x = 5, y = 2; x = 5, y = -2.$$

$$(ii) \text{ Given } x^2 - 7x + 9yi = y^2i + 20i - 12$$

$$\Rightarrow (x^2 - 7x) + i(9y) = (-12) + i(y^2 + 20)$$

$$\Rightarrow x^2 - 7x = -12 \text{ and } 9y = y^2 + 20$$

$$\Rightarrow x^2 - 7x + 12 = 0 \text{ and } y^2 - 9y + 20 = 0$$

$$\Rightarrow (x - 4)(x - 3) = 0 \text{ and } (y - 5)(y - 4) = 0$$

$$\Rightarrow x = 4, 3 \text{ and } y = 5, 4.$$

Hence, the required values of  $x$  and  $y$  are

$$x = 4, y = 5; x = 4, y = 4; x = 3, y = 5; x = 3, y = 4.$$

**Example 5.** Express each of the following in the standard form  $a + ib$  :

$$(i) \left( \frac{1}{3} + i\frac{7}{3} \right) + \left( 4 + i\frac{1}{3} \right) - \left( -\frac{4}{3} + i \right)$$

$$(ii) 3(7 + i7) + i(7 + i7)$$

(NCERT)

(NCERT)

$$(iii) (-2 + \sqrt{-3})(-3 + 2\sqrt{-3})$$

$$(iv) \frac{(3 + i\sqrt{5})(3 - i\sqrt{5})}{(\sqrt{3} + \sqrt{2}i) - (\sqrt{3} - i\sqrt{2})}.$$

(NCERT)

$$\textbf{Solution.} (i) \left( \frac{1}{3} + i\frac{7}{3} \right) + \left( 4 + i\frac{1}{3} \right) - \left( -\frac{4}{3} + i \right)$$

$$= \left( \frac{1}{3} + i\frac{7}{3} \right) + \left( 4 + i\frac{1}{3} \right) + \left( \frac{4}{3} - i \right)$$

$$= \left( \frac{1}{3} + 4 + \frac{4}{3} \right) + i \left( \frac{7}{3} + \frac{1}{3} - 1 \right) = \frac{17}{3} + \frac{5}{3}i.$$

$$(ii) 3(7 + i7) + i(7 + i7) = (21 + 21i) + (7i + 7i^2)$$

$$= 21 + 21i + 7i + 7(-1) = (21 - 7) + (21 + 7)i$$

$$= 14 + 28i.$$

$$(iii) (-2 + \sqrt{-3})(-3 + 2\sqrt{-3}) = (-2 + \sqrt{3}i)(-3 + 2\sqrt{3}i)$$

$$= (6 - 2\sqrt{3}\sqrt{3}) + (-3\sqrt{3} - 4\sqrt{3})i$$

$$= 0 - 7\sqrt{3}i.$$

$$\begin{aligned}
 (iv) \quad \frac{(3 + i\sqrt{5})(3 - i\sqrt{5})}{(\sqrt{3} + \sqrt{2}i) - (\sqrt{3} - i\sqrt{2})} &= \frac{(3)^2 + (\sqrt{5})^2}{\sqrt{2}i + \sqrt{2}i} & ((a + ib)(a - ib) = a^2 + b^2) \\
 &= \frac{9 + 5}{2\sqrt{2}i} = \frac{14}{2\sqrt{2}} \cdot \frac{1}{i} = \frac{7}{\sqrt{2}}(-i) & \left(\because \frac{1}{i} = -i\right) \\
 &= 0 - \frac{7}{\sqrt{2}}i.
 \end{aligned}$$

**Example 6.** Express the following in the form  $a + ib$  :

$$\begin{aligned}
 (i) \quad (-i)(2i) \left(-\frac{1}{8}i\right)^3 & \text{ (NCERT)} & (ii) \quad i^{102} & (iii) \quad i^{-39} & \text{ (NCERT)} \\
 (iv) \quad (-\sqrt{-1})^{31} & (v) \quad i^9 + i^{19} & \text{ (NCERT)} & (vi) \quad i^{35} + \frac{1}{i^{35}}.
 \end{aligned}$$

**Solution.**

$$\begin{aligned}
 (i) \quad (-i)(2i) \left(-\frac{1}{8}i\right)^3 &= (-1)^4 \times 2 \times \left(\frac{1}{8}\right)^3 \times i^5 \\
 &= 1 \times 2 \times \frac{1}{512} \times i^4 \times i \\
 &= \frac{1}{256} \times 1 \times i = 0 + \frac{1}{256}i. \\
 (ii) \quad i^{102} &= i^{4 \times 25 + 2} = i^2 & (\because i^{4k+2} = i^2, k \in \mathbf{I}) \\
 &= -1 = -1 + i0. \\
 (iii) \quad i^{-39} &= i^{4 \times (-10) + 1} = i & (\because i^{4k+1} = i, k \in \mathbf{I}) \\
 &= 0 + i. \\
 (iv) \quad (-\sqrt{-1})^{31} &= (-i)^{31} = (-1)^{31} i^{31} \\
 &= -i^{4 \times 7 + 3} = -i^3 & (\because i^{4k+3} = i^3, k \in \mathbf{I}) \\
 &= -i^2 \cdot i = -(-1)i = i = 0 + i. \\
 (v) \quad i^9 + i^{19} &= i^{2 \times 4 + 1} + i^{4 \times 4 + 3} = i + i^3 \\
 &= i + i^2 \cdot i = i + (-1)i = 0 = 0 + i0. \\
 (vi) \quad i^{35} + \frac{1}{i^{35}} &= i^{35} + i^{-35} = i^{4 \times 8 + 3} + i^{4 \times (-9) + 1} \\
 &= i^3 + i = i^2 i + i = (-1)i + i \\
 &= 0 = 0 + i0.
 \end{aligned}$$

**Example 7.** Express each of the following in the standard form  $a + ib$  :

$$\begin{aligned}
 (i) \quad (1 - i)^4 & \text{ (NCERT)} & (ii) \quad \left(-2 - \frac{1}{3}i\right)^3 & \text{ (NCERT)} \\
 (iii) \quad (2i - i^2)^2 + (1 - 3i)^3 & & (iv) \quad \left(i^{18} + \left(\frac{1}{i}\right)^{25}\right)^3 & \text{ (NCERT)} \\
 (v) \quad (1 + i)^6 + (1 - i)^3 & & & \text{ (NCERT Exemplar Problems)}
 \end{aligned}$$

**Solution.**

$$\begin{aligned}
 (i) \quad (1 - i)^4 &= ((1 - i)^2)^2 = (1 + i^2 - 2i)^2 \\
 &= (1 + (-1) - 2i)^2 = (-2i)^2 = 4i^2 \\
 &= 4(-1) = -4 = -4 + i0.
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \left(-2 - \frac{1}{3}i\right)^3 &= (-1)^3 \left(2 + \frac{1}{3}i\right)^3 \\
 &= -\left[2^3 + 3 \times 2^2 \times \frac{1}{3}i + 3 \times 2 \times \left(\frac{1}{3}i\right)^2 + \left(\frac{1}{3}i\right)^3\right] \\
 &= -\left[8 + 4i + \frac{2}{3}i^2 + \frac{1}{27}i^3\right] \\
 &= -\left[8 + 4i + \frac{2}{3}(-1) + \frac{1}{27}(-i)\right] \\
 &= -\left[\frac{22}{3} + \frac{107}{27}i\right] = -\frac{22}{3} - \frac{107}{27}i.
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad (2i - i^2)^2 + (1 - 3i)^3 &= (2i + 1)^2 + (1 - 3i)^3 \\
 &= (4i^2 + 4i + 1) + (1 - 9i + 27i^2 - 27i^3) \\
 &= -4 + 4i + 1 + 1 - 9i - 27 + 27i = -29 + 22i.
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad \left(i^{18} + \left(\frac{1}{i}\right)^{25}\right)^3 &= \left(i^{4 \times 4 + 2} + (-i)^{25}\right)^3 & \left(\because \frac{1}{i} = -i\right) \\
 &= \left(i^2 + (-1)^{25}i^{25}\right)^3 = (-1 - i^{4 \times 6 + 1})^3 \\
 &= (-1 - i)^3 = (-1)^3(1 + i)^3 \\
 &= -[1 + 3i + 3i^2 + i^3] \\
 &= -[1 + 3i - 3 - i] = -(-2 + 2i) \\
 &= 2 - 2i.
 \end{aligned}$$

$$\begin{aligned}
 \text{(v)} \quad (1 + i)^6 &= ((1 + i)^2)^3 = (1 + i^2 + 2i)^3 = (1 - 1 + 2i)^3 = (2i)^3 \\
 &= 8i^3 = 8(-i) = -8i
 \end{aligned}$$

$$\begin{aligned}
 \text{and} \quad (1 - i)^3 &= 1 - i^3 - 3i + 3i^2 = 1 - (-i) - 3i + 3(-1) \\
 &= -2 - 2i
 \end{aligned}$$

$$\therefore (1 + i)^6 + (1 - i)^3 = -8i + (-2 - 2i) = -2 - 10i.$$

**Example 8.** Find the multiplicative inverse of  $\sqrt{5} + 3i$ .

(NCERT)

**Solution.** Let  $z = \sqrt{5} + 3i$ ,

then  $\bar{z} = \sqrt{5} - 3i$  and  $|z|^2 = (\sqrt{5})^2 + 3^2 = 5 + 9 = 14$ .

We know that the multiplicative inverse of  $z$  is given by

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{\sqrt{5} - 3i}{14} = \frac{\sqrt{5}}{14} - \frac{3}{14}i.$$

**Alternatively**

$$\begin{aligned}
 z^{-1} &= \frac{1}{z} = \frac{1}{\sqrt{5} + 3i} = \frac{1}{\sqrt{5} + 3i} \times \frac{\sqrt{5} - 3i}{\sqrt{5} - 3i} \\
 &= \frac{\sqrt{5} - 3i}{(\sqrt{5})^2 - (3i)^2} = \frac{\sqrt{5} - 3i}{5 - 9(-1)} = \frac{\sqrt{5} - 3i}{14} = \frac{\sqrt{5}}{14} - \frac{3}{14}i.
 \end{aligned}$$

**Example 9.** Express the following in the form  $a + ib$  :

$$\text{(i)} \quad \frac{i}{1+i}$$

$$\text{(ii)} \quad \frac{5 + \sqrt{2}i}{1 - \sqrt{2}i}$$

(NCERT)

$$\text{(iii)} \quad \left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right) \quad (\text{NCERT})$$

$$\text{(iv)} \quad \frac{(1+i)(3+i)}{3-i} - \frac{(1-i)(3-i)}{3+i}.$$

**Solution.** (i)  $\frac{i}{1+i} = \frac{i}{1+i} \times \frac{1-i}{1-i} = \frac{i-i^2}{1-i^2} = \frac{i-(-1)}{1-(-1)} = \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2}i.$

(ii)  $\frac{5 + \sqrt{2}i}{1 - \sqrt{2}i} = \frac{5 + \sqrt{2}i}{1 - \sqrt{2}i} \times \frac{1 + \sqrt{2}i}{1 + \sqrt{2}i} = \frac{5 + 5\sqrt{2}i + \sqrt{2}i + 2i^2}{1^2 - (\sqrt{2}i)^2}$   
 $= \frac{5 + 6\sqrt{2}i - 2}{1 - 2(-1)} = \frac{3 + 6\sqrt{2}i}{3} = 1 + 2\sqrt{2}i.$

(iii)  $\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right) = \frac{1+i-2+8i}{(1-4i)(1+i)} \times \frac{3-4i}{5+i}$   
 $= \frac{-1+9i}{1+i-4i+4} \times \frac{3-4i}{5+i} = \frac{(-1+9i)(3-4i)}{(5-3i)(5+i)}$   
 $= \frac{-3+4i+27i+36}{25+5i-15i+3} = \frac{33+31i}{28-10i}$   
 $= \frac{33+31i}{28-10i} \times \frac{28+10i}{28+10i} = \frac{33 \times 28 + 330i + 31 \times 28i - 310}{(28)^2 - (10i)^2}$   
 $= \frac{924 - 310 + (330 + 868)i}{784 - 100(-1)} = \frac{614 + 1198i}{884} = \frac{307}{442} + \frac{599}{442}i.$

(iv)  $\frac{(1+i)(3+i)}{3-i} - \frac{(1-i)(3-i)}{3+i} = \frac{(1+i)(3+i)(3+i) - (1-i)(3-i)(3-i)}{(3-i)(3+i)}$   
 $= \frac{(1+i)(8+6i) - (1-i)(8-6i)}{9-i^2}$   
 $= \frac{(2+14i) - (2-14i)}{9+1} = \frac{28i}{10} = 0 + \frac{14}{5}i.$

**Example 10.** (i) If  $\frac{(1+i)^2}{2-i} = x + iy$ , then find the value of  $x + y$ . (NCERT Exemplar Problems)

(ii) If  $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x + iy$ , then find  $(x, y)$ . (NCERT Exemplar Problems)

(iii) If  $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$ , then find  $(a, b)$ . (NCERT Exemplar Problems)

**Solution.** (i)  $x + iy = \frac{(1+i)^2}{2-i} = \frac{1+i^2+2i}{2-i} = \frac{1-1+2i}{2-i} = \frac{2i}{2-i}$   
 $= \frac{2i}{2-i} \times \frac{2+i}{2+i} = \frac{4i+2i^2}{2^2-i^2} = \frac{4i+2(-1)}{4-(-1)}$   
 $= \frac{-2+4i}{5} = -\frac{2}{5} + \frac{4}{5}i.$

$$\Rightarrow x = -\frac{2}{5} \text{ and } y = \frac{4}{5}.$$

$$\therefore x + y = -\frac{2}{5} + \frac{4}{5} = \frac{2}{5}.$$

(ii) We have,  $\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{(1+i)^2}{1^2-i^2}$   
 $= \frac{1+i^2+2i}{1-(-1)} = \frac{1-1+2i}{2} = \frac{2i}{2} = i$

$$\therefore \frac{1-i}{1+i} = \frac{1}{i} = -i \quad \left(\because \frac{1}{i} = -i\right)$$

$$\begin{aligned}\text{Given } x + iy &= \left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = i^3 - (-i)^3 = i^3 + i^3 \\ &= 2i^3 = 2(-i) = 0 - 2i\end{aligned}$$

$$\Rightarrow x = 0 \text{ and } y = -2.$$

Hence, the ordered pair  $(x, y)$  is  $(0, -2)$ .

$$\begin{aligned}\text{(iii) Given } a + ib &= \left(\frac{1-i}{1+i}\right)^{100} = (-i)^{100} && \text{(see part (ii))} \\ &= (-1)^{100} (i)^{4 \times 25} = 1 \times 1 = 1 = 1 + 0i\end{aligned}$$

$$\Rightarrow a = 1 \text{ and } b = 0.$$

Hence, the ordered pair  $(a, b)$  is  $(1, 0)$ .

**Example 11.** (i) If  $(1 + i)z = (1 - i)\bar{z}$ , then show that  $z = -i\bar{z}$ . (NCERT Exemplar Problems)

$$\text{(ii) If } z_1 = 2 - i \text{ and } z_2 = -2 + i, \text{ find } \operatorname{Re} \left( \frac{z_1 z_2}{\bar{z}_1} \right). \quad \text{(NCERT)}$$

**Solution.** (i) Given  $(1 + i)z = (1 - i)\bar{z}$

$$\Rightarrow \frac{z}{\bar{z}} = \frac{1-i}{1+i} = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = \frac{(1-i)^2}{1^2 - i^2} = \frac{1 + i^2 - 2i}{1 - (-1)}$$

$$\Rightarrow \frac{z}{\bar{z}} = \frac{1-1-2i}{2} = -\frac{2i}{2} = -i$$

$$\Rightarrow z = -i\bar{z}.$$

$$\begin{aligned}\text{(ii)} \quad \frac{z_1 z_2}{\bar{z}_1} &= \frac{(2-i)(-2+i)}{\overline{2-i}} = \frac{-4+2i+2i-i^2}{2+i} = \frac{-4+4i-(-1)}{2+i} \\ &= \frac{-3+4i}{2+i} = \frac{-3+4i}{2+i} \times \frac{2-i}{2-i} \\ &= \frac{-6+3i+8i-4i^2}{2^2-i^2} = \frac{-6+11i-4(-1)}{4-(-1)} \\ &= \frac{-2+11i}{5} = -\frac{2}{5} + \frac{11}{5}i.\end{aligned}$$

$$\therefore \operatorname{Re} \left( \frac{z_1 z_2}{\bar{z}_1} \right) = -\frac{2}{5}.$$

**Example 12.** (i) Find the conjugate of  $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$  (NCERT)

(ii) Find the modulus of  $\frac{1+i}{1-i} - \frac{1-i}{1+i}$  (NCERT)

(iii) Find the modulus of  $\frac{(2-3i)^2}{-1+5i}$

$$\begin{aligned}\text{Solution. (i) Let } z &= \frac{(3-2i)(2+3i)}{(1+2i)(2-i)} = \frac{6+9i-4i+6}{2-i+4i+2} \\ &= \frac{12+5i}{4+3i} = \frac{12+5i}{4+3i} \times \frac{4-3i}{4-3i} = \frac{48-36i+20i+15}{4^2-(3i)^2} \\ &= \frac{63-16i}{16-9(-1)} = \frac{63-16i}{25} = \frac{63}{25} - \frac{16}{25}i.\end{aligned}$$

$$\therefore \text{Conjugate of } z = \frac{63}{25} + \frac{16}{25}i.$$

**Example 32.** If  $\alpha$  and  $\beta$  are different complex numbers with  $|\beta| = 1$ , then find  $\left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right|$ .  
(NCERT)

**Solution.** We have  $\left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right| = \left| \frac{(\beta - \alpha)\bar{\beta}}{(1 - \bar{\alpha}\beta)\bar{\beta}} \right|$  (Note this step)

$$= \left| \frac{(\beta - \alpha)\bar{\beta}}{\bar{\beta} - \bar{\alpha}\beta\bar{\beta}} \right| = \left| \frac{(\beta - \alpha)\bar{\beta}}{\bar{\beta} - \bar{\alpha}} \right| \quad (\text{Given } |\beta| = 1 \Rightarrow |\beta|^2 = 1 \Rightarrow \beta\bar{\beta} = 1)$$

$$= \frac{|(\beta - \alpha)\bar{\beta}|}{|\bar{\beta} - \bar{\alpha}|} \quad \left( \because \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \right)$$

$$= \frac{|\beta - \alpha| |\bar{\beta}|}{|\bar{\beta} - \bar{\alpha}|} \quad (\because \overline{z_1 - z_2} = \overline{z_1} - \overline{z_2})$$

$$= \frac{|\beta - \alpha| |\beta|}{|\bar{\beta} - \bar{\alpha}|} \quad (\because |\bar{z}| = |z|)$$

$$= |\beta| = 1 \quad (|\beta| = 1, \text{ given})$$

Hence,  $\left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right| = 1$ .

**Example 33.** If  $|z_1| = |z_2| = \dots = |z_n| = 1$ , prove that

$$|z_1 + z_2 + \dots + z_n| = \left| \frac{1}{z_1} + \frac{1}{z_2} + \dots + \frac{1}{z_n} \right|. \quad (\text{NCERT Exemplar Problems})$$

**Solution.** Given  $|z_1| = |z_2| = \dots = |z_n| = 1$

$$\Rightarrow |z_1|^2 = |z_2|^2 = \dots = |z_n|^2 = 1$$

$$\Rightarrow z_1 \bar{z}_1 = 1, z_2 \bar{z}_2 = 1, \dots, z_n \bar{z}_n = 1$$

$$\Rightarrow \frac{1}{z_1} = \bar{z}_1, \frac{1}{z_2} = \bar{z}_2, \dots, \frac{1}{z_n} = \bar{z}_n.$$

$$\therefore \left| \frac{1}{z_1} + \frac{1}{z_2} + \dots + \frac{1}{z_n} \right| = |\bar{z}_1 + \bar{z}_2 + \dots + \bar{z}_n|$$

$$= |\overline{z_1 + z_2 + \dots + z_n}| = |z_1 + z_2 + \dots + z_n|. \quad (\because |\bar{z}| = |z|)$$

## EXERCISE 5.1

*Very short answer type questions (1 to 31) :*

1. Evaluate the following :

(i)  $\sqrt{-9} \times \sqrt{-4}$

(ii)  $\sqrt{(-9)(-4)}$

(iii)  $\sqrt{-25} \times \sqrt{16}$

(iv)  $3\sqrt{-16}\sqrt{-25}$

(v)  $\sqrt{-16} + 3\sqrt{-25} + \sqrt{-36} - \sqrt{-625}$ .

2. If  $z = -3 - i$ , find  $\operatorname{Re}(z)$ ,  $\operatorname{Im}(z)$ ,  $\bar{z}$  and  $|z|$ .

3. If  $z^2 = -i$ , then is it true that  $z = \pm \frac{1}{\sqrt{2}}(1 - i)$ ?

4. If  $z^2 = -3 + 4i$ , then is it true that  $z = \pm(1 + 2i)$ ?

5. If  $i = \sqrt{-1}$ , then show that  $(x + 1 + i)(x + 1 - i) = x^2 + 2x + 2$ .

6. Find real values of  $x$  and  $y$  if

(i)  $2y + (3x - y)i = 5 - 2i$

(ii)  $(3x - 1) + (\sqrt{3} + 2y)i = 5$

(iii)  $(3y - 2) + i(7 - 2x) = 0$ .

7. If  $x, y \in \mathbf{R}$  and  $(5y - 2) + i(3x - y) = 3 - 7i$ , find the values of  $x$  and  $y$ .
8. If  $x, y$  are reals and  $(3y + 2) + i(x + 3y) = 0$ , find the values of  $x$  and  $y$ .
9. If  $x, y$  are reals and  $(1 - i)x + (1 + i)y = 1 - 3i$ , find the values of  $x$  and  $y$ .
10. For any two complex numbers  $z_1$  and  $z_2$ , prove that  

$$\operatorname{Re}(z_1 z_2) = \operatorname{Re}(z_1) \operatorname{Re}(z_2) - \operatorname{Im}(z_1) \operatorname{Im}(z_2). \quad (\text{NCERT})$$
11. For any complex number  $z$ , prove that  

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2} \text{ and } \operatorname{Im}(z) = \frac{z - \bar{z}}{2i}.$$
12. If  $z$  is a complex number, show that  $\frac{z - \bar{z}}{2i}$  is real.  
 Express the following (13 to 20) complex numbers in the standard form  $a + ib$  :
13. (i)  $(-5i)\left(\frac{1}{8}i\right)$  (NCERT) (ii)  $(5i)\left(-\frac{3}{5}i\right)$ . (NCERT)
14. (i)  $(1 - i) - (-1 + 6i)$  (NCERT) (ii)  $\left(\frac{1}{5} + i\frac{2}{5}\right) - \left(4 + i\frac{5}{2}\right)$ . (NCERT)
15. (i)  $(-2 + 3i) + 3\left(-\frac{1}{2}i + 1\right) - (2i)$  (ii)  $(7 + i5)(7 - i5)$ .
16. (i)  $(-\sqrt{3} + \sqrt{-2})(2\sqrt{3} - i)$  (NCERT) (ii)  $(-5 + 3i)^2$ .
17. (i)  $\left(\frac{1}{2} + 2i\right)^3$  (ii)  $(5 - 3i)^3$ . (NCERT)
18. (i)  $\left(\frac{1}{3} + 3i\right)^3$  (NCERT) (ii)  $(\sqrt{5} + 7i)(\sqrt{5} - 7i)^2$ .
19. (i)  $i^{99}$  (ii)  $i^{-35}$ . (NCERT)
20. (i)  $(-\sqrt{-4})^3$  (ii)  $i + i^2 + i^3 + i^4$ .
21. Find the value of  $(-1 + \sqrt{-3})^2 + (-1 - \sqrt{-3})^2$ .
22. If  $n$  is any integer, then find the value of  
 (i)  $(-\sqrt{-1})^{4n+3}$  (ii)  $\frac{i^{4n+1} - i^{4n-1}}{2}$ . (NCERT Exemplar Problems)
23. Find the multiplicative inverse of  $-i$ . (NCERT)
24. Express the following numbers in the form  $a + ib$ ,  $a, b \in \mathbf{R}$  :  
 (i)  $\frac{i}{1+i}$  (ii)  $\frac{1-i}{1+i}$ .
25. If  $(a + ib)(c + id) = A + iB$ , then show that  $(a^2 + b^2)(c^2 + d^2) = A^2 + B^2$ .
26. Find the modulus of the following complex numbers :  
 (i)  $(3 - 4i)(-5 + 12i)$  (ii)  $\frac{5 - 12i}{-3 + 4i}$ .
27. Find the modulus of the following :  
 (i)  $\frac{(2 - 3i)^2}{4 + 3i}$  (ii)  $(\sqrt{7} - 3i)^3$ .
28. (i) If  $z = 3 - \sqrt{7}i$ , then find  $|z^{-1}|$ .  
 (ii) If  $z = x + iy$ ,  $x, y \in \mathbf{R}$ , then find  $|iz|$ .
29. Find the conjugate of  $i^7$ .
30. Write the conjugate of  $(2 + 3i)(1 - 2i)$  in the form  $a + ib$ ,  $a, b \in \mathbf{R}$ .
31. Solve for  $x$  :  $|1 + i|^x = 2$ .  
 Express the following (32 and 33) complex numbers in the standard form  $a + ib$  :
32. (i)  $i^{55} + i^{60} + i^{65} + i^{70}$  (ii)  $\frac{i + i^2 + i^4}{1 + i^2 + i^4}$ .

44. If  $a + ib = \frac{(x + i)^2}{2x^2 + 1}$ , then prove that  $a^2 + b^2 = \frac{(x^2 + 1)^2}{(2x^2 + 1)^2}$ .

45. If  $z = 1 + 2i$ , then find the value of  $z^3 + 7z^2 - z + 16$ .

46. Show that if  $\left| \frac{z - 5i}{z + 5i} \right| = 1$ , then  $z$  is a real number.

47. (i) If  $z = x + iy$  and  $\left| \frac{z - 2}{z - 3} \right| = 2$ , show that  $3(x^2 + y^2) - 20x + 32 = 0$ .

(NCERT Exemplar Problems)

(ii) If  $z = x + iy$  and  $\frac{|z - 1 - i| + 4}{3|z - 1 - i| - 2} = 1$ , show that  $x^2 + y^2 - 2x - 2y = 7$ .

48. Find the least positive integral value of  $n$  for which  $\left( \frac{1+i}{1-i} \right)^n$  is a real number.

49. Find the real value of  $\theta$  such that  $\frac{1 + i \cos \theta}{1 - 2i \cos \theta}$  is a real number.

50. If  $z$  is a complex number such that  $|z| = 1$ , prove that  $\frac{z-1}{z+1}$  ( $z \neq -1$ ) is purely imaginary.  
What is the exception?

(NCERT Exemplar Problems)

51. If  $z_1, z_2$  and  $z_3$  are complex numbers such that  $|z_1| = |z_2| = |z_3| = \left| \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right| = 1$ , then find the value of  $|z_1 + z_2 + z_3|$ .

(NCERT Exemplar Problems)

## 5.2 ARGAND PLANE

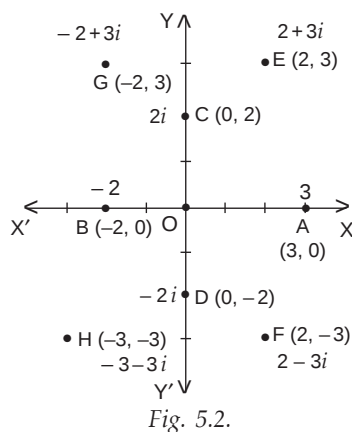
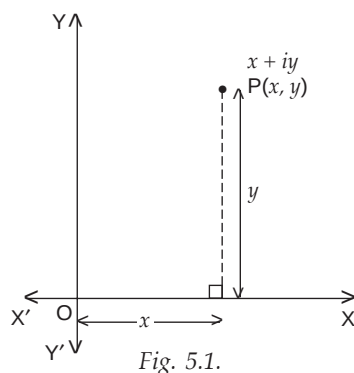
We know that corresponding to every real number there exists a unique point on the number line (called real axis) and conversely corresponding to every point on the line there exists a unique real number *i.e.* there is a one-one correspondence between the set  $\mathbf{R}$  of real numbers and the points on the real axis.

In a similar way, corresponding to every ordered pair  $(x, y)$  of real numbers there exists a unique point  $P$  in the co-ordinate plane with  $x$  as **abscissa** and  $y$  as **ordinate** of the point  $P$  and conversely corresponding to every point  $P$  in the plane there exists a unique ordered pair of real numbers. Thus, there is a one-one correspondence between the set of ordered pairs  $\{(x, y); x, y \in \mathbf{R}\}$  and the points in the co-ordinate plane.

The point  $P$  with co-ordinates  $(x, y)$  is said to represent the complex number  $z = x + iy$ .

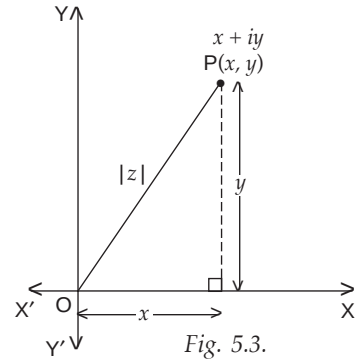
It follows that the complex number  $z = x + iy$  can be uniquely represented by the point  $P(x, y)$  in the co-ordinate plane and conversely corresponding to the point  $P(x, y)$  in the plane there exists a unique complex number  $z = x + iy$ . The co-ordinate plane that represents the complex numbers is called the **complex plane** or **Argand plane**.

The complex numbers  $3, -2, 2i, -2i, 2 + 3i, 2 - 3i, -2 + 3i$  and  $-3 - 3i$  which correspond to the ordered pairs  $(3, 0), (-2, 0), (0, 2), (0, -2), (2, 3), (2, -3), (-2, 3)$  and  $(-3, -3)$  respectively have been represented geometrically in the co-ordinate plane by the points A, B, C, D, E, F, G and H respectively in fig. 5.2.



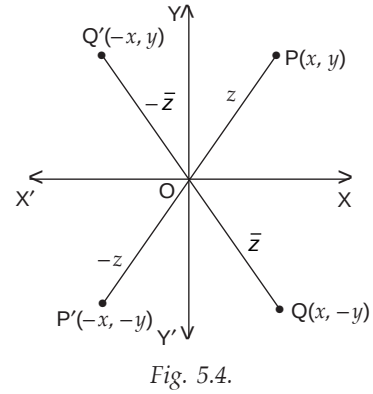
Note that every real number  $x = x + 0i$  is represented by point  $(x, 0)$  lying on  $x$ -axis, and every purely imaginary number  $iy$  is represented by point  $(0, y)$  lying on  $y$ -axis. Consequently,  $x$ -axis is called the **real axis** and  $y$ -axis is called the **imaginary axis**.

If the point  $P(x, y)$  represents the complex number  $z = x + iy$ , then the distance between the points  $P$  and the origin  $O(0, 0) = \sqrt{x^2 + y^2} = |z|$ . Thus, the modulus of  $z$  i.e.  $|z|$  is the distance between points  $P$  and  $O$  (see fig. 5.3).



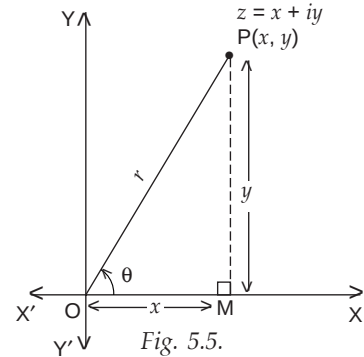
**Geometric representation of  $-z$ ,  $\bar{z}$  and  $-\bar{z}$ .** If  $z = x + iy$ ,  $x, y \in \mathbf{R}$  is represented by the point  $P(x, y)$  in the complex plane, then the complex numbers  $-z$ ,  $\bar{z}$ ,  $-\bar{z}$  are represented by the points  $P'(-x, -y)$ ,  $Q(x, -y)$  and  $Q'(-x, y)$  respectively in the complex plane (see fig. 5.4).

Geometrically, the point  $Q(x, -y)$  is the mirror image of the point  $P(x, y)$  in the real axis. Thus, conjugate of  $z$  i.e.  $\bar{z}$  is the mirror image of  $z$  in the  $x$ -axis.



### 5.3 POLAR REPRESENTATION OF COMPLEX NUMBERS

Let the point  $P(x, y)$  represent the non-zero complex number  $z = x + iy$  in the Argand plane. Let the directed line segment  $OP$  be of length  $r (> 0)$  and  $\theta$  be the radian measure of the angle which  $OP$  makes with the positive direction of  $x$ -axis (shown in fig. 5.5). Then  $r = \sqrt{x^2 + y^2} = |z|$  and is called **modulus** of  $z$ ; and  $\theta$  is called **amplitude** or **argument** of  $z$  and is written as **amp(z)** or **arg(z)**.

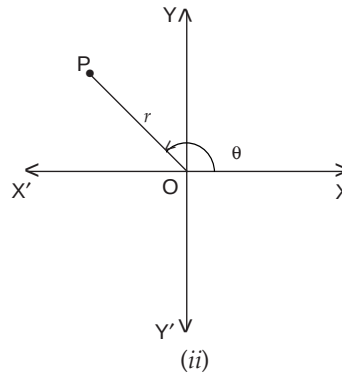
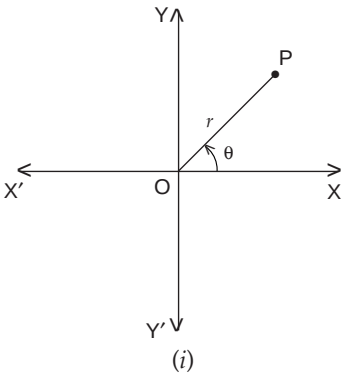


From figure 5.5, we see that

$$x = r \cos \theta \text{ and } y = r \sin \theta$$

$$\therefore z = x + iy = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta).$$

Thus,  $z = r(\cos \theta + i \sin \theta)$ . This form of  $z$  is called **polar form** of the complex number  $z$ .



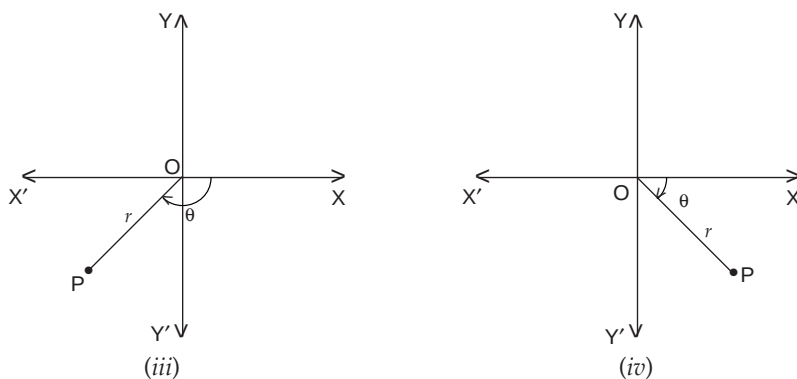


Fig. 5.6.

For any non-zero complex number  $z$ , there corresponds only one value of  $\theta$  in  $-\pi < \theta \leq \pi$  (see fig. 5.6). The unique value of  $\theta$  such that  $-\pi < \theta \leq \pi$  is called **principal value of amplitude** or **argument**.

Thus every (non-zero) complex number  $z = x + iy$  can be uniquely expressed as  $z = r(\cos \theta + i \sin \theta)$  where  $r > 0$  and  $-\pi < \theta \leq \pi$  and conversely, for every  $r > 0$  and  $\theta$  such that  $-\pi < \theta \leq \pi$ , we get a unique (non-zero) complex number  $z = r(\cos \theta + i \sin \theta) = x + iy$ .

Note that the complex number zero cannot be put into the form  $r(\cos \theta + i \sin \theta)$  and so, the argument of zero complex number does not exist.

#### REMARK

If we take origin as the pole and the positive direction of the  $x$ -axis as the initial line, then the point  $P$  is uniquely determined by the ordered pair of real numbers  $(r, \theta)$ , called the **polar co-ordinates** of the point  $P$  (see fig. 5.6).

### ILLUSTRATIVE EXAMPLES

**Example 1.** Convert the following complex numbers in the polar form and represent them in Argand plane :

- |                            |                              |                                |
|----------------------------|------------------------------|--------------------------------|
| (i) $\sqrt{3} + i$ (NCERT) | (ii) $-\sqrt{3} + i$ (NCERT) | (iii) $-1 - i\sqrt{3}$ (NCERT) |
| (iv) $2 - 2i$              | (v) $-3$ (NCERT)             | (vi) $-5i$ .                   |

**Solution.** (i) Let  $z = \sqrt{3} + i = r(\cos \theta + i \sin \theta)$ .

Then  $r \cos \theta = \sqrt{3}$  and  $r \sin \theta = 1$ .

On squaring and adding, we get

$$r^2 (\cos^2 \theta + \sin^2 \theta) = (\sqrt{3})^2 + 1^2$$

$$\Rightarrow r^2 = 4 \Rightarrow r = 2.$$

$$\therefore \cos \theta = \frac{\sqrt{3}}{2} \text{ and } \sin \theta = \frac{1}{2}.$$

The value of  $\theta$  such that  $-\pi < \theta \leq \pi$  and satisfying both the above equations is given by  $\theta = \frac{\pi}{6}$ .

Hence,  $z = 2 \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$ , which is the required polar form.

The complex number  $z = \sqrt{3} + i$  is represented in fig. 5.7.

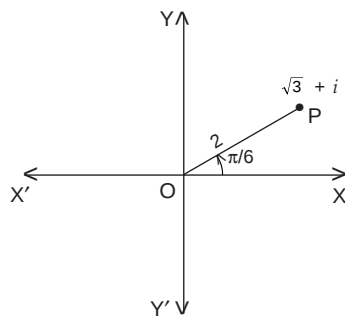


Fig. 5.7.

**ANSWERS****EXERCISE 5.1**

1. (i)  $-6$  (ii)  $6$  (iii)  $20i$  (iv)  $-60$  (v)  $0$   
 2.  $-3$ ;  $-1$ ;  $-3 + i$ ;  $\sqrt{10}$ . 3. Yes 4. Yes  
 6. (i)  $x = \frac{1}{6}$ ,  $y = \frac{5}{2}$  (ii)  $x = 2$ ,  $y = -\frac{\sqrt{3}}{2}$  (iii)  $x = \frac{7}{2}$ ,  $y = \frac{2}{3}$   
 7.  $x = -2$ ,  $y = 1$  8.  $x = 2$ ,  $y = -\frac{2}{3}$  9.  $x = 2$ ,  $y = -1$   
 13. (i)  $\frac{5}{8} + i0$  (ii)  $3 + i0$  14. (i)  $2 - 7i$  (ii)  $-\frac{19}{5} - \frac{21}{10}i$   
 15. (i)  $1 - \frac{1}{2}i$  (ii)  $74 + i0$  16. (i)  $(-6 + \sqrt{2}) + \sqrt{3}(1 + 2\sqrt{2})i$  (ii)  $16 - 30i$   
 17. (i)  $-\frac{47}{8} - \frac{13}{2}i$  (ii)  $-10 - 198i$  18. (i)  $-\frac{242}{27} - 26i$  (ii)  $54\sqrt{5} - 378i$   
 19. (i)  $0 - i$  (ii)  $0 + i$  20. (i)  $0 + 8i$  (ii)  $0 + i0$   
 21.  $-4$  22. (i)  $i$  (ii)  $i$  23.  $i$   
 24. (i)  $\frac{1}{2} + \frac{1}{2}i$  (ii)  $0 - i$  26. (i)  $65$  (ii)  $\frac{13}{5}$   
 27. (i)  $\frac{13}{5}$  (ii)  $64$  28. (i)  $\frac{1}{4}$  (ii)  $\sqrt{x^2 + y^2}$   
 29.  $i$  30.  $8 + i$  31.  $2$   
 32. (i)  $0 + i0$  (ii)  $0 + i$  33. (i)  $1 - i$  (ii)  $16 + i0$   
 34. (i)  $\frac{2}{13} + \frac{3}{13}i$  (ii)  $\frac{4}{25} + \frac{3}{25}i$  (iii)  $\frac{3}{16} - \frac{\sqrt{7}}{16}i$   
 35. (i)  $\frac{21}{25} - \frac{47}{25}i$  (ii)  $-\frac{1}{4} - \frac{\sqrt{3}}{4}i$  (iii)  $\frac{2}{5} + \frac{29}{5}i$   
 (iv)  $\frac{8}{65} + \frac{1}{65}i$  (v)  $\frac{1}{2} + \frac{1}{2}i$  (vi)  $\frac{63}{25} - \frac{16}{25}i$   
 36. (i)  $\frac{40}{41} - \frac{9}{41}i$ ;  $\frac{40}{41} + \frac{9}{41}i$ ;  $1$  (ii)  $1 + i$ ;  $1 - i$ ;  $\sqrt{2}$   
 (iii)  $-1 + i$ ;  $-1 - i$ ;  $\sqrt{2}$  (iv)  $-9 + 46i$ ;  $-9 - 46i$ ;  $\sqrt{2197}$   
 37. (i)  $0 + \frac{1}{2}i$  (iii)  $1$  (iv)  $2^n$  38. (i)  $\frac{11}{5}$  (ii)  $\frac{1}{5}$  (iii)  $0$   
 39. (i)  $x = \frac{5}{13}$ ,  $y = \frac{14}{13}$  (ii)  $x = 6$ ,  $y = 1$  (iii)  $x = \frac{2}{21}$ ,  $y = -\frac{8}{21}$   
 40. (i)  $\frac{3}{2} - 2i$  is the only solution (ii) all purely imaginary numbers  
 45.  $-17 + 24i$  48.  $2$   
 49.  $(2n + 1)\frac{\pi}{2}$ ,  $n \in \mathbb{I}$  50. Exception is  $z = 1$  51.  $1$

**EXERCISE 5.2**

1. (i) True (ii) True (iii) True (iv) True (v) True  
 2.  $0$  3.  $-\theta$   
 4. (i)  $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$  (ii)  $1 - i$  (iii)  $0 - 3i$  (iv)  $\frac{5}{2} + \frac{5\sqrt{3}}{2}i$  5.  $-2\sqrt{3} + 2i$