# 7

# PERMUTATIONS AND COMBINATIONS

## INTRODUCTION

In daily life, we come across many problems of finding the number of ways of arranging or selecting objects.

Suppose you have a suitcase with a number lock. The number lock has three wheels each marked with 10 digits from 0 to 9. The lock can be opened if 3 specific digits are arranged in a particular order with no repetition of digits. Somehow, you have forgotten this particular order of the digits. The question arises! 'How many arrangements you have to check with to open the lock?' To answer this question, you may immediately start listing all the possible arrangements of 10 digits taken 3 at a time with no repetitions. But, this method of listing the arrangements is very tedious and time consuming, because the number of possible arrangements is large.

In the present chapter, we will learn some basic techniques of counting which will enable us to answer the above question without actually listing 3-digit arrangements. In fact, these techniques will be useful in determining the number of different ways of arranging and selecting objects in a wide variety of situations.

## 7.1 FUNDAMENTAL PRINCIPLE OF COUNTING

Let us first consider the following examples.

**1.** Sania has 2 school bags and 3 water bottles. In how many different ways can she carry these objects to her school choosing one each?

As there are 2 school bags, a school bag can be chosen in 2 different ways. Similarly, a water bottle can be chosen in 3 different ways. For every choice of a school bag, there are three choices of a water bottle, therefore, the number of different ways in which a pair of a school bag and a water bottle can be chosen = 2 × 3 = 6. Hence Sania can carry these objects to her school choosing one each in 6 different ways.

If we represent the 2 school bags by $S_1$, $S_2$ and the 3 water bottles by $W_1$, $W_2$, $W_3$, then the six different ways can be shown by the following tree diagram :

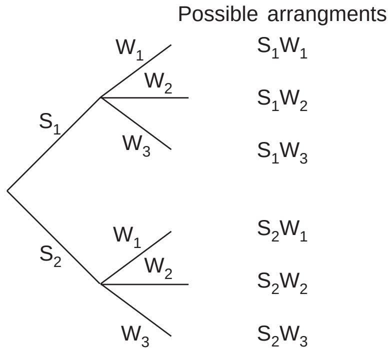


*Fig. 7.1.*

In the above diagram, the first set of branches shows the two possible choices of school bags and the second set of branches shows the three possible choices of water bottles corresponding to each choice of a school bag. The different arrangements are $S_1W_1$, $S_1W_2$, $S_1W_3$, $S_2W_1$, $S_2W_2$ and $S_2W_3$.

**2.** Numbers 1, 2, 3 and 4 are written on four cards. How many 2-digit numbers can be formed by placing two cards side by side?

We may pick up any one of the four cards and put it down representing the digit as ten's place. After this, pick up any card out of the remaining 3 cards and put it to the right of the previous card representing a digit at unit's place. For every choice of a card at ten's place, there are 3 choices for a card at unit's place, therefore, the number of different numbers which can be formed by placing two cards side by side = 4 × 3 = 12.

Look at the following tree diagram :

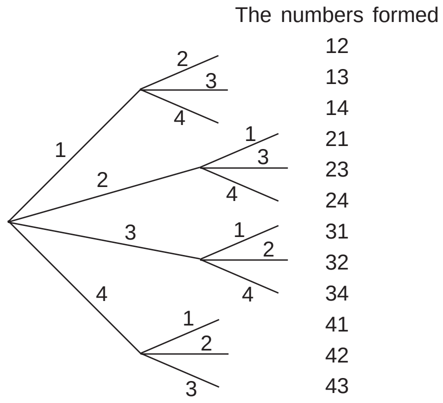


*Fig. 7.2.*

In the above diagram, the first set of branches shows the four possible choices of cards, placed at ten's place and the second set of branches shows the three possible choices of cards at unit's place corresponding to each choice of cards placed as ten's place. Two digit numbers formed are :

12, 13, 14,    21, 23, 24,    31, 32, 34,    41, 42, 43

The above examples lead to the following principle of counting :

## Fundamental principle of counting

*If an event can occur in m different ways, and if when it has occurred, a second event can occur in n different ways, then the total number of different ways of occurrence of the two events is m × n.*

*The above principle is also called* **multiplication principle of counting.**

In example 1, the required number of ways of carrying a school bag and a water bottle was the number of ways of the occurrence of the following events in succession :

(*i*) the event of choosing a school bag

(*ii*) the event of choosing a water bottle.

The above principle of counting can be generalised to any finite number of events. Now, let us consider the following example :

Gopal has 8 shirts, 5 pants and 2 pairs of shoes. In how many different ways can he dress up himself with?

As Gopal has 8 shirts, he can choose a shirt in 8 ways. Similarly, he can choose a pant in 5 ways. For every choice of a shirt, there are 5 choices of a pant. Therefore, a shirt and a pant can be chosen in 8 × 5 ways *i.e.* 40 ways. Further, as he has 2 pairs of shoes, he can choose a pair of shoes in 2 ways. For every choice of a shirt and a pant, there are 2 choices of a pair of shoes. Therefore, a shirt, a pant and a pair of shoes (complete dress) can be chosen in 40 × 2 ways *i.e.* 80 ways. Hence Gopal can dress up himself in 80 different ways.

The above example leads to the following generalisation of the principle of counting :

*If an event can occur in m different ways, and if when it has occurred, a second event can occur in n different ways, following which a third event can occur in p different ways and so forth, then the total number of different ways of occurrence of all the events is* $m \times n \times p \ldots$ .

## ILLUSTRATIVE EXAMPLES

**Example 1.** *(i) You can go from Delhi to Jaipur either by car or by bus or by train or by air. In how many ways can you plan your journey from Delhi to Jaipur and back to Delhi?*

*(ii) Suppose you like variety and you don't want to return by the same mode of transport. How many different ways are possible?*

**Solution.** (*i*) You can go from Delhi to Jaipur in 4 different ways, and you can return from Jaipur to Delhi in 4 different ways. So, by multiplication principle of counting, there are 4 × 4 = 16 different ways of going from Delhi to Jaipur and returning to Delhi.

(*ii*) You can go from Delhi to Jaipur in 4 different ways. Corresponding to each of these, there are 3 different ways of returning from Jaipur to Delhi (since a different mode of transport has to be chosen while returning). Hence, the required number of ways is 4 × 3 = 12.

**Example 2.** *(i) A lady wants to select one cotton saree and one polyester saree from a textile shop. If there are 10 cotton varieties and 12 polyester varieties, in how many ways can she choose the two sarees ?*

*(ii) If in above case, the lady has limited budget, in how many ways can she choose one saree?*

**Solution.** (*i*) The lady can select one cotton saree out of 10 cotton varieties in 10 ways since any of 10 varieties can be selected. Corresponding to each selection of a cotton saree, she can choose a polyester saree in 12 ways. Hence the two sarees (one cotton and one polyester), by multiplication principle of counting, can be selected in 10 × 12 = 120 ways.

(*ii*) Since only one saree is to be chosen and 10 + 12 = 22 different varieties of sarees are available, one saree (either of cotton or of polyester) can be chosen in 22 ways.

**Example 3.** *In a class, there are 27 boys and 14 girls. In how many ways can the teacher form a team of one boy and one girl from amongst the students of the class to represent the class for a function?* (*NCERT Examplar Problems*)

**Solution.** The teacher can select one boy out of 27 boys in 27 ways since any of 27 boys can be selected. Corresponding to each selection of a boy, the teacher can select one girl in 14 ways since any of 14 girls can be selected.

Hence a team consisting of one boy and one girl, by multiplication principle of counting, can be formed in (27 × 14) ways *i.e.* in 378 ways.

**Example 4.** *Three persons enter a railway carriage, where there are 5 vacant seats. In how many ways can they seat themselves ?*

**Solution.** First man can sit on any of 5 vacant seats. Then the second can sit on any of 4 vacant seats left. And the third can sit on any of 3 vacant seats left. Hence by fundamental principle of counting, the required number of ways = 5 × 4 × 3 = 60.

**Example 5.** *How many words can be formed out of the letters of the word 'MAGIC' taking all the letters at a time (no letter being repeated) ?*

**Solution.** There are 5 different letters with which 5 vacant places □ □ □ □ □ are to be filled up. The first place can be filled in 5 ways, as any one of the five letters M, A, G, I, C can be placed there. Having filled up the first place in any of the 5 ways, 4 letters are left and any one of them can be placed in second place. Hence the first two places can together be filled up in 5 × 4 ways. Now three letters are left and any of them can be put in third place. After that, 2 letters are left and any of them can be put in fourth place. After that only 1 letter is left and it has to be placed in fifth place. Therefore, the total number of ways of filling up five places = 5 × 4 × 3 × 2 × 1 = 120.

Hence, the required number of words formed = 120.

**Example 6.** *How many 4 letter code words are possible using the first 10 letters of the English alphabet if*

*(i) no letter can be repeated? (NCERT) (ii) letters may be repeated?*

**Solution.** (*i*) No letter can be repeated.

There are 10 different ways for choosing the first letter of the code, 9 different ways for choosing the second letter of the code, 8 ways for choosing the third letter and 7 ways for choosing the fourth letter of the code.

Therefore, the total number of 4 letter code words

$$= 10 \times 9 \times 8 \times 7 = 5040.$$

(*ii*) Letters can be repeated.

There are 10 different ways for choosing the first letter of the code. As the letters can be repeated, there are 10 different ways for choosing the second letter of the code.

Similarly, there are 10 ways of choosing for each of the third and the fourth letter of the code.

Therefore, the total number of 4 letter code words

$$= 10 \times 10 \times 10 \times 10 = 10000.$$

**Example 7.** *How many automobile license plates can be made if each plate contains two different letters followed by three different digits?* (*NCERT Examplar Problems*)

**Solution.** The number of ways of forming license plates containing two different letters followed by three different digits is the same as filling five vacant places $\overset{\text{I}}{\square}\ \overset{\text{II}}{\square}\ \overset{\text{III}}{\square}\ \overset{\text{IV}}{\square}\ \overset{\text{V}}{\square}$, first two places by two different letters and the next three places by three different digits.

The first place can be filled in 26 ways by any of 26 letters and the second place can be filled in 25 ways by any of 25 remaining letters. The third place can be filled in 10 ways by any of the digits from 0 to 9, the fourth place can be filled in 9 ways by any of the remaining 9 digits and the fifth place can be filled in 8 ways by any of the remaining 8 digits.

By multiplication principle of counting, the total number of ways of filling these places *i.e.* the total number of ways of forming license plates = 26 × 25 × 10 × 9 × 8 = 468000.

**Example 8.** *In a certain city, all telephone numbers have six digits, the first two digits always being 41 or 42 or 46 or 62 or 64. How many telephone numbers have all six digits distinct?*

(*NCERT Examplar Problems*)

**Solution.** The number of telephone numbers of six digits having all six distinct digits and the first two digits being 41 or 42 or 46 or 62 or 64 is the same as filling six vacant places $\underbrace{\overset{\text{I}}{\square}\overset{\text{II}}{\square}}\overset{\text{III}}{\square}\overset{\text{IV}}{\square}\overset{\text{V}}{\square}\overset{\text{VI}}{\square}$, first two places by any of the numbers 41, 42, 46, 62 or 64 and the remaining four places by the other digits, of course, all six digits distinct.

We note that the first two places (*i.e.* I and II) together can be filled in 5 ways.

As two different digits are already used in first place, so we are left with only 8 different digits.

Third place can be filled in 8 ways by any of 8 remaining digits.

Similarly, fourth place can be filled in 7 ways, fifth in 6 ways and sixth place in 5 ways.

By multiplication principle of counting, the number of ways of filling all the six places (under given condition) = 5 × 8 × 7 × 6 × 5.

∴ Required number of telephone numbers = 5 × 8 × 7 × 6 × 5 = 8400.

**Example 9.** *How many 3-digit even numbers can be formed from the digits 1, 2, 3, 4, 5, 6 if the digits can be repeated?* *(NCERT)*

**Solution.** There are six different digits with which 3 vacant places □ □ □ are to be filled up satisfying the given conditions.

As the required 3-digit numbers are even, we start filling in unit's place and the options for this place are 2, 4, and 6 only. So unit's place can be filled up in 3 different ways. Having filled up unit's place, the ten's place can be filled up by any of the six digits 1, 2, 3, 4, 5 and 6 in six different ways as the digits can be repeated. Similarly, the hundred's place can be filled up in 6 ways.

Therefore, by the multiplication principle of counting, the required number of 3-digit even numbers = 3 × 6 × 6 = 108.

**Example 10.** *How many numbers lying between 100 and 1000 can be formed with the digits 0, 1, 2, 3, 4, 5 if the repetition of digits is not allowed?* *(NCERT)*

**Solution.** The numbers lying between 100 and 1000 consist of 3 digits. There are six different digits with which three vacant places □ □ □ are to be filled up, repetition of digits not allowed.

As the required numbers are of 3-digits, the first place (hundred's place) has to be non-zero so the options for this place are 1, 2, 3, 4, 5. So hundred's place can be filled up in 5 different ways. Having filled up hundred's place, the second place (ten's place) can be filled up in 5 ways with any one of the remaining 5 digits as the repetition of digits is not allowed. The third place (unit's place) can be filled up in 4 ways with any one of the remaining 4 digits.

By fundamental principle of counting, the required number of numbers lying between 100 and 1000 = 5 × 5 × 4 = 100.

**Example 11.** *How many 4-digit numbers are there with no digit repeated?* *(NCERT)*

**Solution.** There are 10 different digits (0 to 9) with which four vacant places □ □ □ □ are to be filled up, repetition of digits not allowed. As the required numbers are of 4-digits, the first place (thousand's place) has to be non-zero so there are 9 choices. So the first place can be filled up in 9 ways. Having filled up the first place, the second place (hundred's place) can be filled up in 9 ways by any of the remaining 9 digits as the repetition of digits is not allowed.

Similarly, the third place (ten's place) can be filled up in 8 ways and the fourth place (unit's place) can be filled up in 7 ways.

By fundamental principle of counting, the required number of 4-digit numbers
= 9 × 9 × 8 × 7 = 4536.

**Example 12.** *Find the number of 3-digit odd numbers, when repetition of digits is allowed.*

**Solution.** There are 10 different digits (0 to 9) with which three vacant places □ □ □ are to be filled up satisfying the given conditions, repetition of digits is allowed.

As the required 3-digit numbers are odd, we start filling in unit's place and the options for this place are 1, 3, 5, 7 and 9 only. So unit's place can be filled up in 5 different ways. Having filled up unit's place, the ten's place can be filled up by any of the 10 digits (0 to 9) in 10 ways. As the hundred's place has to be non-zero, so this place can be filled up in 9 different ways. By fundamental principle of counting, the required number of 3-digit odd numbers
= 5 × 10 × 9 = 450.

**Example 13.** *Find the number of 4-digit odd numbers, when repetition of digits is not allowed.*

**Solution.** There are 10 different digits (0 to 9) with which four vacant places □ □ □ □ are to be filled up satisfying the given conditions, repetition of digits is not allowed.

As the required 4-digit numbers are odd, we start filling in unit's place and the options for this place are 1, 3, 5, 7 and 9 only. So unit's place can be filled up in 5 different ways. After filling unit's place, we fill thousand's place. As the thousand's place has to be non-zero and the repetition of digits is not allowed, so this place can be filled by the remaining 8 digits (one digit is used for unit's place and 0 cannot be filled at this place) in 8 different ways.

After filling unit's place and thousand's place, we fill up the remaining two places *i.e.* hundred's place and ten's place.

As two digits are already used and repetition of digits is not allowed, therefore, the hundred's place can be filled up by the remaining 8 digits in 8 ways and the ten's place can be filled up by the remaining 7 digits in 7 ways.

By fundamental principle of counting, the required number of 4-digit odd numbers when repetition of digits not allowed = 5 × 8 × 8 × 7 = 2240.

**Example 14.** *Find the number of positive integers greater than 6000 and less than 7000 which are divisible by 5, provided that no digit is to be repeated.* (*NCERT Examplar Problems*)

**Solution.** We are to find the number of positive integers between 6000 and 7000 which are divisible by 5. So, we are to fill up four vacant places □ □ □ □ by 10 different digits (0 to 9) satisfying the given conditions and repetition of digits is not allowed.

The thousand's place has to be filled up by the digit 6, so there is only one way to fill up this place. As the numbers are divisible by 5, so the unit's place has to be filled up either by 5 or by 0. Thus, there are only 2 ways to fill up unit's place.

Now, the hundred's place can be filled up in 8 ways by any of the remaining 8 digits because at thousand's place and at unit's place two digits have already been used.

Similarly, the ten's place can be filled up in 7 ways.

∴ The number of required positive integers = 1 × 8 × 7 × 2 = 112.

**Example 15.** *In how many ways can 5 persons sit in a car, 2 including the driver in the front seat and 3 in the back seat, if 2 particular persons do not know driving?*

**Solution.** Let us mark the 5 seats by the letters A, B, C, D and E with A as driver's seat.

Since 2 particular persons out of the 5 do not know driving, there are 3 choices for seat A, 4 choices for seat B, 3 choices for seat C, 2 choices for seat D and one choice for seat E.

Therefore, the total number of seating arrangements

= 3 × 4 × 3 × 2 × 1 = 72.

**Example 16.** *Given 5 flags of different colours. How many different signals can be generated by hoisting the flags on a vertical pole (one below the other) if each signal requires the use of*

(*i*) *two flags?* (*NCERT*) (*ii*) *atleast two flags?* (*NCERT*)

**Solution.** (*i*) There will be as many different signals as the different number of ways of filling in two vacant places (one above the other) by 5 flags of different colours. The upper vacant place can be filled in 5 different ways by any one of the 5 flags. Having filled up the upper place in any one of the 5 ways, 4 flags are left. The lower place can be filled in by any one of the remaining 4 flags. Therefore, by multiplication principle of counting, the number of different signals that can be generated by hoisting two flags (one below the other) = 5 × 4 = 20.

(*ii*) As a signal consists of atleast 2 flags, so a signal can consist of either 2 flags, 3 flags, 4 flags or 5 flags. Let us count the possible number of signals generated by hoisting 2 flags, 3 flags, 4 flags and 5 flags separately (one below the other).

The number of different signals generated by hoisting 2 flags

= 5 × 4 = 20. (see (*i*))

Similarly, the number of different signals generated by hoisting 3 flags

= 5 × 4 × 3 = 60.

The number of different signals generated by hoisting 4 flags

= 5 × 4 × 3 × 2 = 120.

The number of different signals generated by hoisting 5 flags

= 5 × 4 × 3 × 2 × 1 = 120.

Therefore, the number of different signals generated by hoisting atleast two flags

= 20 + 60 + 120 + 120 = 320.

**Example 17.** *How many numbers can be formed from the digits 1, 2, 3, 9 if the repetition of digits is not allowed ?*

**Solution.** We have to consider 1-digit numbers, 2-digit numbers, 3-digit numbers and 4-digit numbers.

Here we are given four digits 1, 2, 3 and 9; repetition of digits not allowed.

Number of 1-digit numbers = 4.

Let us consider 2-digit numbers. The first place (ten's place) can be filled in 4 ways, then second place (unit's place) can be filled in 3 ways. Therefore, number of 2-digit numbers $= 4 \times 3 = 12$.

Similarly, number of 3-digit numbers $= 4 \times 3 \times 2 = 24$.

The number of 4-digit numbers $= 4 \times 3 \times 2 \times 1 = 24$.

Hence, the required number of numbers $= 4 + 12 + 24 + 24 = 64$.

**Example 18.** *How many numbers are there between 100 and 1000 (including 100 but excluding 1000) such that*

*(i) every digit is either 2 or 5* *(ii) there is no restriction*

*(iii) no digit is repeated* *(iv) the digit in hundred's place is 5*

*(v) the digit in unit's place is 5* *(vi) atleast one of the digits is 5*

*(vii) atleast one of the digits is repeated* *(viii) exactly one digit is 5 ?*

**Solution.** The numbers between 100 and 1000 consist of 3 digits — 100 is included and 1000 is excluded.

(*i*) Since every digit is 5 or 2, there are 2 ways of filling up of each of three digits (places). Thus the three digits (places) can be filled in $2 \times 2 \times 2 = 8$ ways. Hence required number of numbers = 8.

(*ii*) The first digit (hundred's place) has to be non-zero so there are 9 choices. Second and third digits can be any of ten digits 0 to 9. Hence required number of numbers $= 9 \times 10 \times 10 = 900$ (including 100 but excluding 1000).

(*iii*) First digit (hundred's place) can be any digit from 1 to 9; second digit can be then chosen in 9 ways, and third digit in 8 ways. Thus required number of numbers with no digit repeated $= 9 \times 9 \times 8 = 648$.

(*iv*) Digit in hundreds place is fixed (5). The other two places can each be filled in 10 ways. Hence the possible ways are $1 \times 10 \times 10 = 100$.

(*v*) The digit in hundred's place can be filled in 9 ways (1 to 9); the digit in tens place can be filled in 10 ways (0 to 9), and digit in units place can be filled in one way only as it is fixed (5). Hence the required number of numbers $= 9 \times 10 \times 1 = 90$.

(*vi*) First, let us find all numbers between 100 and 1000 which don't have digit 5 in any place. The number of such numbers is $8 \times 9 \times 9 = 648$ (including 100). We have also seen in part (*ii*) that there are 900 numbers between 100 and 1000 (including 100). Hence there are $900 - 648 = 252$ numbers between 100 and 1000 which have atleast one digit as 5.

(*vii*) As there are 900 numbers between 100 and 1000 (including 100) and 648 of them have no digit repeated, so there are $900 - 648 = 252$ numbers (including 100) which have atleast one digit repeated.

(*viii*) The numbers with exactly one digit as 5 are :

(*a*) of the type $5 \times \times = 1 \times 9 \times 9 = 81$ (where × is a non-five digit)

(*b*) of the type $\times 5 \times = 8 \times 1 \times 9 = 72$ (first digit is non-zero non-five)

(*c*) of the type $\times \times 5 = 8 \times 9 \times 1 = 72$.

Thus, there are $81 + 72 + 72 = 225$ numbers between 100 and 1000 which have exactly one digit as 5.

**Example 19.** *How many of the natural numbers from 1 to 1000 have none of their digits repeated ?*

**Solution.** We have to consider natural numbers consisting of 1-digit, 2-digits and 3-digits, and find how many of these have distinct digits. Obviously, we are ignoring 1000 as it has 0 as repeated digit.

Number of 1-digit natural numbers = 9 (as it can be any one of 1, 2, 3, …, 9).

Let us consider 2-digit numbers. The ten's place has to be non-zero, so there are 9 choices. As no digit is to be repeated, there are 9 choices for unit's place. Hence the number of 2-digit numbers = $9 \times 9 = 81$.

Similarly, the number of 3-digit numbers = $9 \times 9 \times 8 = 648$.

Therefore, the number of numbers from 1 to 1000 which have none of their digits repeated

$$= 9 + 81 + 648 = 738.$$

**Example 20.** *Find the total of natural numbers that can be formed by using the digits 0, 1, 2, 3, 4 and 5, repetition of digits is not allowed.*

**Solution.** We are required to form natural numbers consisting of 1-digit, 2-digits, 3-digits, 4-digits, 5-digits and 6-digits from the digits 0, 1, 2, 3, 4 and 5, without repetition of digits.

Number of 1-digit natural numbers = 5.

Let us form 2-digit natural numbers from the given digits.

The ten's place has to be non-zero, so there are 5 choices (1, 2, 3, 4, 5) for this place. As no digit is to be repeated, there are 5 choices for unit's place.

∴ The number of 2-digit natural numbers = $5 \times 5 = 25$.

Similarly, the number of 3-digit natural numbers = $5 \times 5 \times 4 = 100$.

The number of 4-digit natural numbers = $5 \times 5 \times 4 \times 3 = 300$.

The number of 5-digit natural numbers = $5 \times 5 \times 4 \times 3 \times 2 = 600$.

The number of 6-digit natural numbers = $5 \times 5 \times 4 \times 3 \times 2 \times 1 = 600$.

Therefore, the number of required natural numbers

$$= 5 + 25 + 100 + 300 + 600 + 600 = 1630.$$

**Example 21.** *How many odd numbers less than 1000 can be formed using the digits 0, 1, 4, 5, 7, 8 if the repetition of digits is allowed?*

**Solution.** The numbers less than 1000 will consist of 1-digit, 2-digits or 3-digits. The digits to be used are 0, 1, 4, 5, 7, 8 satisfying the given condition and repetition of digits is allowed.

***1-digit odd numbers***

Number of 1-digit odd numbers = 3 (1, 5 or 7)

***2-digit odd numbers***

The number of ways of filling up unit's place = 3 (1, 5 or 7)

As the ten's place has to be non-zero, the number of ways of filling up ten's place

= 5 (1, 4, 5, 7 or 8).

∴ The number of 2-digit odd numbers = $3 \times 5 = 15$.

***3-digit odd numbers***

The number of ways of filling up unit's place = 3 (1, 5 or 7)

The number of ways of filling up ten's place = 6 (0, 1, 4, 5, 7 or 8).

The number of ways of filling up hundred's place = 5 (1, 4, 5, 7 or 8).

∴ The number of 3-digit odd numbers = $3 \times 6 \times 5 = 90$.

∴ The total number of the required numbers = $3 + 15 + 90 = 108$.

**Example 22.** *How many odd numbers less than 10000 can be formed using the digits 0, 1, 4, 5, 7, 8 if the repetition of digits is not allowed?*

**Solution.** The numbers less than 10000 will consist of 1-digit, 2-digits, 3-digits or 4-digits. The digits to be used are 0, 1, 4, 5, 7, 8 satisfying the given condition and repetition of digits not allowed.

***1-digit odd numbers***

Number of 1-digit odd numbers = 3 (1, 5 or 7)

***2-digit odd numbers***

The number of ways of filling up unit's place = 3 (1, 5 or 7).

As the ten's place has to non-zero and repetition of digits not allowed, the number of ways of filling up ten's place = 4 (because one non-zero digit is already used at unit's place).

$\therefore$ The number of 2-digit odd numbers = 3 × 4 = 12.

***3-digit odd numbers***

The number of ways of filling up units place = 3 (1, 5 or 7).

After filling unit's place, we fill hundred's place. As the hundred's place has to be non-zero and repetition of digits not allowed, the number of ways of filling hundred's place = 4.

The number of ways of fillig up ten's place = 4.

$\therefore$ The number of 3-digit odd numbers = 3 × 4 × 4 = 48.

***4-digit odd numbers***

The number of ways of filling up unit's place = 3 (1, 5 or 7).

After filling unit's place, we fill thousand's place. As this place has to be non-zero and repetition of digits not allowed, the number of ways of filling up thousand's place = 4.

The number of ways of filling up hundred's place = 4.

The number of ways of filling up ten's place = 3.

$\therefore$ The number of 4-digit odd numbers = 3 × 4 × 4 × 3 = 144.

$\therefore$ The total number of required numbers = 3 + 12 + 48 + 144 = 207.

**Example 23.** *In how many ways can 5 different balls be distributed among 3 boxes ?*

**Solution.** Each ball can be put into any one of the three boxes in 3 different ways.

Therefore, by fundamental principle of counting, the number of ways of distributing 5 different balls among three boxes

$$= 3 \times 3 \times 3 \times 3 \times 3 = 3^5 = 243.$$

**Example 24.** *Find the total number of ways of answering 6 multiple choice questions, each question having 4 choices.*

**Solution.** Each question can be answered in 4 ways.

Therefore, by fundamental principle of counting, the number of ways of answering 6 multiple choice questions

$$= 4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^6.$$

## EXERCISE 7.1

***Very short answer type questions (1 to 20) :***

**1.** In a cricket match, how many choices can be made for man of the match ?

**2.** In a class there are 22 girls and 17 boys. The teacher wants to select either a girl or a boy to represent the class in a function. In how many ways can the teacher make this selection?

**3.** In a class there are 17 girls and 22 boys. In how many ways can the teacher form a team of one girl and one boy from amongst the students of the class to represent the school in a quiz competition?

**4.** Of 11 cricket players, one is to be chosen as captain and another as vice captain. How many choices are there?

**5.** Lata wants to go abroad by air and return by ship. She has a choice of 6 different airlines to go and 4 different ships to return. In how many ways she can perform her journey ?

6. There are 4 doors in Lotus temple. In how many ways can a person enter the temple and leave by a different door ?

7. Four persons A, B, C and D are to give lectures to an audience. In how many ways can the organiser arrange the order of their presentation?

8. Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word ROSE (repetition of letters not allowed). (*NCERT*)

9. For a group photograph, 3 boys and 2 girls stand in a line in all possible ways. How many photos could be taken if each photo corresponds to each such arrangement?

10. Six pictures are to be arranged (in line from left to right) on a wall of an art gallery for display. How many arrangements are possible ?

11. Sandy has 5 shirts, 4 pants, 3 pairs of socks and 2 pairs of shoes. In how many different ways can he dress himself ?

12. How many 3-letter code words are possible using the first 10 letters of English alphabet if

    (*i*) no letter can be repeated ? (*ii*) letters can be repeated ?

13. How many different 5-letter code words can be formed using the first 10 letters of the English alphabet, if no letter can be repeated?

14. How many outcomes are possible when a coin is tossed

    (*i*) 2 times? (*ii*) 3 times? (*NCERT*) (*iii*) 4 times? (*iv*) 5 times?

15. Given 6 flags of different colours. How many different signals can be generated by hoisting the flags on a vertical pole (one below the other) if each signal requires the use of

    (*i*) 2 flags? (*ii*) 3 flags? (*iii*) 4 flags?

16. How many 3-digit numbers can be formed from the digits 1, 2, 3, 4 and 5 if

    (*i*) repetition of digits is allowed ? (*ii*) repetition of digits is not allowed ? (*NCERT*)

17. How many four digit numbers can be formed in which all the digits are different?

18. How many three digit numbers can be formed without using the digits 2, 3, 5, 6, 7 and 9?

19. How many three digit numbers can be formed using the digits 0, 1, 3, 5, 6, 7 if

    (*i*) repetition of digits is not allowed? (*ii*) repetition of digits is allowed?

20. How many 2-digit even numbers can be formed from the digits 1, 2, 3, 4 and 5 if the digits

    (*i*) can repeat (*NCERT*) (*ii*) cannot repeat?

21. How many 3-digit odd numbers can be formed from the digits 1, 2, 3, 4 and 5 if

    (*i*) repetition of digits is allowed ? (*ii*) repetition of digits is not allowed ?

22. How many 3-digit even numbers can be made by using the digits 1, 2, 3, 4, 6 and 7 if

    (*i*) repetition of digits is not allowed ? (*NCERT*)
    (*ii*)repetition of digits is allowed ?

23. How many 3-digit odd numbers can be formed using the digits 0, 1, 2, 3, 4 and 5 if

    (*i*) repetition of digits is allowed? (*ii*) repetition of digits is not allowed?

24. How many numbers between 99 and 1000 (both excluding) can be formed such that

    (*i*) every digit is either 3 or 7? (*ii*) there is no restriction?
    (*iii*) no digit is repeated? (*iv*) the digit in hundred's place is 7?
    (*v*) the digit in ten's place is 7? (*vi*) the digit 7 does not appear at any place?
    (*vii*) the digit in unit's place is 7? (*NCERT Examplar Problems*)
    (*viii*) atleast one of the digits is 7? (*NCERT Examplar Problems*)

**25.** What is the sum of the digits in the units place of all the numbers formed using the digits 3, 4, 5, 6 taken all at a time?

**Hint.** $(3 + 4 + 5 + 6) \times \underline{|3}$.

**26.** In how many ways can 4 different balls be distributed in 3 boxes?

**27.** In how many ways can 3 letters be posted in 4 letter boxes?

**28.** In how many ways can 3 letters be posted in 4 letters boxes so that all the letters are not posted in the same letter box?

**29.** In how many ways 4 different balls be distributed in 5 boxes so that all the balls are not put in the same box ?

**30.** Vijeta has 6 friends to invite for her son's birthday party. In how many ways can she send invitation cards to them if she has three servants to carry the cards?

**31.** There are 12 buses running between Delhi and Agra. In how many ways can a man plan his journey from Delhi to Agra and back ? In how many ways can he go from Delhi to Agra and return by a different bus ?

**32.** How many words (with or without meaning) of three English alphabets can be formed ? How many of these have all distinct alphabets ?

**33.** In how many ways can 2 prizes (in Science and Maths) be awarded to 15 students ? In how many ways can the first and second prize in History be awarded to 15 students ?

**34.** Two cards are drawn, one at a time, and without replacement, from a deck of 52 cards. Determine the number of ways in which cards can be drawn. What will be the number of ways if the first card is replaced before the second is drawn ?

**35.** Eight children are to be seated on a bench.

(*i*) In how many ways can the children be seated ?

(*ii*) How many arrangements are possible if the youngest child sits at the left hand end of the bench?

**36.** A number lock on a suitcase has 3 wheels each labelled with ten digits from 0 to 9. If the opening of the lock is a particular sequence of three digits with no repeats, how many such sequences are possible?

**37.** How many 5-digit telephone numbers can be constructed using the digits 0 to 9 if each number starts with 67 and no digit appears more than once ? (*NCERT*)

**38.** How many different code words are there such that two distinct English alphabets are followed by two distinct non-zero digits? How many of them end in an odd integer ?

**39.** A class consists of 60 boys and 40 girls. In how many ways can a president, vice president, secretary and treasurer can be chosen if the secretary must be a boy, the treasurer must be a girl and a student may not hold more than one office?

**Hint.** First choose a secretary and a treasurer. Any one of 60 boys can be chosen as secretary and any one of 40 girls can be chosen as treasurer. Now, we are left with 59 boys and 39 girls *i.e.* 98 students. Any one of these students can be chosen as president, following which a vice president can be chosen in 97 ways.

**40.** Given 6 flags of different colours. How many different signals can be generated by hoisting the flags on a vertical pole (one below the other) if each signal requires the use of atleast 4 flags?

**41.** How many numbers are there between 100 and 1000 such that atleast one of their digits is 3 ?

**42.** How many 5-digit even numbers can be formed from the digits 0, 1, 2, 3 and 4, if repetition of digits is not allowed?

**43.** How many numbers divisible by 5 and lying between 4000 and 5000 can be formed from the digits 2, 4, 5, 7 and 9, if

(*i*) repetition of digits is allowed (*ii*) repetition of digits is not allowed?

**44.** How many numbers divisible by 5 and lying between 40000 and 50000 can be formed from the digits 0, 3, 4, 5, 8 and 9, if

(*i*) repetition of digits is not allowed (*ii*) repetition of digits is allowed?

**45.** How many odd numbers less than 1000 can be formed using the digits 0, 4, 5, 7 if

(*i*) repetition of digits is allowed? (*ii*) repetition of digits is not allowed?

## 7.2 FACTORIAL NOTATION

***Factorial.*** *The continued product of first n natural numbers is called n* ***factorial*** *or* ***factorial*** *n* and is denoted by $\lfloor\underline{n}$ or $n!$

Thus, $\lfloor\underline{n}$ or $n! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1)n$

$= n(n-1)(n-2) \times \ldots \times 3 \times 2 \times 1$ (in reverse order)

For example, $\lfloor\underline{1} = 1$

$\lfloor\underline{2} = 1 \times 2 = 2$

$\lfloor\underline{3} = 1 \times 2 \times 3 = 6$

$\lfloor\underline{4} = 1 \times 2 \times 3 \times 4 = 24$

$\lfloor\underline{5} = 1 \times 2 \times 3 \times 4 \times 5 = 120$ and so on.

Note that $\lfloor\underline{n} = n(n-1)(n-2) \times \ldots \times 3 \times 2 \times 1$

$= n\lfloor\underline{n-1}$ for $n > 1$

$= n(n-1)\lfloor\underline{n-2}$ for $n > 2$

$= n(n-1)(n-2)\lfloor\underline{n-3}$ for $n > 3$ and so on.

For example, $\lfloor\underline{8} = 8 \times \lfloor\underline{7} = 8 \times 7 \times \lfloor\underline{6}$ and so on.

**Meaning of zero factorial.** According to the above definition, $\lfloor\underline{0}$ makes no sense. However, we define $\lfloor\underline{0} = 1$. It will be clear from the subsequent work why we make this convention.

**NOTE**

When $n$ is a negative integer or a fraction, $n$ factorial is not defined.

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Find the value of*

(*i*) $\dfrac{\lfloor\underline{25}}{\lfloor\underline{23}}$ (*ii*) $\dfrac{\lfloor\underline{9}}{\lfloor\underline{6} \times \lfloor\underline{3}}$ (*iii*) $(7-4)!$ (*iv*) $\lfloor\underline{7} - \lfloor\underline{5}$ (NCERT) (*v*) $\lfloor\underline{2} + \lfloor\underline{3}$.

**Solution.** (*i*) $\dfrac{\lfloor\underline{25}}{\lfloor\underline{23}} = \dfrac{25 \times 24 \times \lfloor\underline{23}}{\lfloor\underline{23}} = 25 \times 24 = 600.$

(*ii*) $\dfrac{\lfloor\underline{9}}{\lfloor\underline{6} \times \lfloor\underline{3}} = \dfrac{9 \times 8 \times 7 \times \lfloor\underline{6}}{\lfloor\underline{6} \times \lfloor\underline{3}} = \dfrac{9 \times 8 \times 7}{\lfloor\underline{3}} = \dfrac{9 \times 8 \times 7}{1 \times 2 \times 3} = 84.$

(*iii*) $(7-4)! = 3! = 1 \times 2 \times 3 = 6.$

(*iv*) $\lfloor\underline{7} - \lfloor\underline{5} = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 - 1 \times 2 \times 3 \times 4 \times 5$

$= 5040 - 120 = 4920.$

(*v*) $\lfloor\underline{2} + \lfloor\underline{3} = 1 \times 2 + 1 \times 2 \times 3 = 2 + 6 = 8.$

**Example 2.** *Evaluate* $\frac{n!}{(n-r)!}$ *when*

*(i)* $n = 9, r = 5$ *(NCERT)* *(ii)* $r = 3$.

**Solution.** (*i*) When $n = 9$, $r = 5$,

$$\frac{n!}{(n-r)!} = \frac{9!}{(9-5)!} = \frac{9!}{4!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4!}{4!}$$

$$= 9 \times 8 \times 7 \times 6 \times 5 = 15120.$$

(*ii*) When $r = 3$,

$$\frac{n!}{(n-r)!} = \frac{n!}{(n-3)!} = \frac{n(n-1)(n-2) \times (n-3)!}{(n-3)!}$$

$$= n(n-1)(n-2).$$

**Example 3.** *Evaluate* $\frac{\lfloor \underline{n}}{\lfloor \underline{r}\, \lfloor \underline{n-r}}$ *when*

*(i)* $n = 7, r = 5$ *(ii)* $n = 52, r = 48$ *(iii)* $r = 2$.

**Solution.** (*i*) When $n = 7$, $r = 5$

$$\frac{\lfloor \underline{n}}{\lfloor \underline{r}\, \lfloor \underline{n-r}} = \frac{\lfloor \underline{7}}{\lfloor \underline{5}\, \lfloor \underline{2}} = \frac{7 \times 6 \times \lfloor \underline{5}}{\lfloor \underline{5} \times \lfloor \underline{2}} = \frac{7 \times 6}{\lfloor \underline{2}} = \frac{7 \times 6}{1 \times 2} = 21.$$

(*ii*) When $n = 52$, $r = 48$

$$\frac{\lfloor \underline{n}}{\lfloor \underline{r}\, \lfloor \underline{n-r}} = \frac{\lfloor \underline{52}}{\lfloor \underline{48}\, \lfloor \underline{4}} = \frac{52 \times 51 \times 50 \times 49 \times \lfloor \underline{48}}{\lfloor \underline{48} \times \lfloor \underline{4}}$$

$$= \frac{52 \times 51 \times 50 \times 49}{1 \times 2 \times 3 \times 4} = 270725.$$

(*iii*) When $r = 2$,

$$\frac{\lfloor \underline{n}}{\lfloor \underline{r}\, \lfloor \underline{n-r}} = \frac{\lfloor \underline{n}}{\lfloor \underline{2}\, \lfloor \underline{n-2}} = \frac{n(n-1)\lfloor \underline{n-2}}{\lfloor \underline{2}\, \lfloor \underline{n-2}} = \frac{n(n-1)}{1 \times 2} = \frac{n(n-1)}{2}.$$

**Example 4.** *Convert the following products into factorials :*

*(i)* $5 \times 6 \times 7 \times 8 \times 9$ *(ii)* $2 \times 4 \times 6 \times 8 \times 10 \times 12$.

**Solution.** (*i*) $5 \times 6 \times 7 \times 8 \times 9 = \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9}{1 \times 2 \times 3 \times 4} = \frac{\lfloor \underline{9}}{\lfloor \underline{4}}$.

(*ii*) $2 \times 4 \times 6 \times 8 \times 10 \times 12 = (2 \times 1)(2 \times 2)(2 \times 3)(2 \times 4)(2 \times 5)(2 \times 6)$

$$= 2^6 \times (1 \times 2 \times 3 \times 4 \times 5 \times 6)$$

$$= 2^6 \times \lfloor \underline{6}.$$

**Example 5.** *Find n, if (i)* $\frac{1}{\lfloor \underline{8}} + \frac{1}{\lfloor \underline{9}} = \frac{n}{\lfloor \underline{10}}$ *(NCERT)* *(ii)* $(n+1)! = 12 \times (n-1)!$

**Solution.** (*i*) $\frac{1}{\lfloor \underline{8}} + \frac{1}{\lfloor \underline{9}} = \frac{n}{\lfloor \underline{10}} \Rightarrow \frac{1}{\lfloor \underline{8}} + \frac{1}{9 \times \lfloor \underline{8}} = \frac{n}{10 \times 9 \times \lfloor \underline{8}}$

$$\Rightarrow \quad \frac{1}{\lfloor \underline{8}}\left(1 + \frac{1}{9}\right) = \frac{n}{10 \times 9 \times \lfloor \underline{8}} \Rightarrow 1 + \frac{1}{9} = \frac{n}{90}$$

$$\Rightarrow \quad \frac{10}{9} = \frac{n}{90} \quad \Rightarrow n = 100.$$

(*ii*) $(n+1)! = 12 \times (n-1)!$

$\Rightarrow$ $(n+1) \times n \times (n-1)! = 12 \times (n-1)!$

$\Rightarrow \quad (n + 1) \times n = 12 \;\Rightarrow n^2 + n - 12 = 0$

$\Rightarrow \quad (n - 3)\,(n + 4) = 0 \;\Rightarrow n - 3 = 0$ or $n + 4 = 0$

$\Rightarrow \quad n = 3$ or $n = -4$.

But $n$ cannot be negative as $n\,!$ is not meaningful when $n$ is negative, therefore, $n = 3$.

**Example 6.** *Find n, if* $\dfrac{\lfloor\underline{n}}{\lfloor\underline{2}\,\lfloor\underline{n-2}}$ *and* $\dfrac{\lfloor\underline{n}}{\lfloor\underline{4}\,\lfloor\underline{n-4}}$ *are in the ratio 2 : 1.*

**Solution.** Given $\dfrac{\lfloor\underline{n}}{\lfloor\underline{2}\,\lfloor\underline{n-2}} : \dfrac{\lfloor\underline{n}}{\lfloor\underline{4}\,\lfloor\underline{n-4}} = 2 : 1$

$$\Rightarrow \quad \frac{\lfloor\underline{n}}{\lfloor\underline{2}\,\lfloor\underline{n-2}} \times \frac{\lfloor\underline{4}\,\lfloor\underline{n-4}}{\lfloor\underline{n}} = \frac{2}{1}$$

$$\Rightarrow \quad \frac{4 \times 3 \times \lfloor\underline{2} \times \lfloor\underline{n-4}}{\lfloor\underline{2} \times (n-2) \times (n-3) \times \lfloor\underline{n-4}} = \frac{2}{1}$$

$$\Rightarrow \quad \frac{4 \times 3}{(n-2)(n-3)} = \frac{2}{1} \;\Rightarrow (n - 2)\,(n - 3) = 6$$

$\Rightarrow \quad n^2 - 5n = 0 \;\Rightarrow n\,(n - 5) = 0$

$\Rightarrow \quad n = 0$ or $n = 5$.

But, for $n = 0$, $\lfloor\underline{n-2}$ and $\lfloor\underline{n-4}$ are not meaningful, therefore, $n = 5$.

**Example 7.** *Prove that* $\lfloor\underline{2n} = 1.3.5. \ldots (2n - 1).2^n.\lfloor\underline{n}$ .

**Solution.** $\lfloor\underline{2n} = 1.2.3.4.5.6. \ldots (2n - 1)\,(2n)$

$= [1.3.5. \ldots (2n - 1)]\,[2.4.6. \ldots 2n]$

$= [1.3.5. \ldots (2n - 1)]\,[(2.1)\,(2.2)\,(2.3) \ldots (2.n)]$

$= 1.3.5. \ldots (2n - 1).2^n.(1.2.3 \ldots n)$

$= 1.3.5. \ldots (2n - 1).2^n.\lfloor\underline{n}$, as desired.

## EXERCISE 7.2

*Very short answer type questions (1 to 5) :*

**1.** Find the values of

(*i*) $\lfloor\underline{6}$  (*ii*) $\lfloor\underline{4} + \lfloor\underline{3}$  (*iii*) $\lfloor\underline{9-6}$  (*iv*) $\lfloor\underline{6} - \lfloor\underline{4}$  (*v*) $\lfloor\underline{3} \times \lfloor\underline{5}$

(*vi*) $\dfrac{\lfloor\underline{88}}{\lfloor\underline{86}}$  (*vii*) $\dfrac{\lfloor\underline{12}}{\lfloor\underline{10}\,\lfloor\underline{2}}$  (*viii*) $\dfrac{\lfloor\underline{9} - \lfloor\underline{8}}{\lfloor\underline{7}}$  (*ix*) $\dfrac{\lfloor\underline{n+2}}{\lfloor\underline{n+1}}$.

**2.** Evaluate $\lfloor\underline{n-r}$ when

(*i*) $n = 6\,;\; r = 2$  (*ii*) $n = 9,\; r = 4$.

**3.** Evaluate $\dfrac{n!}{(n-r)!}$ when

(*i*) $n = 6,\; r = 2$  (*NCERT*)  (*ii*) $n = 10,\; r = 4$

(*iii*) $n = 12,\; r = 3$  (*iv*) $r = 2$.

**4.** Evaluate $\dfrac{n!}{r!\,(n-r)!}$ when

(*i*) $n = 5,\; r = 2$  (*NCERT*)  (*ii*) $n = 15,\; r = 12$  (*iii*) $r = 3$.

**5.** Which of the following are true?

(*i*) $4\left(\lfloor\underline{3}\right) = \lfloor\underline{4}$  (*ii*) $3\left(\lfloor\underline{4}\right) = \lfloor\underline{3 \times 4}$  (*iii*) $\lfloor\underline{3} + \lfloor\underline{4} = \lfloor\underline{3+4}$

(*iv*) $\lfloor\underline{4-3} = \lfloor\underline{4} - \lfloor\underline{3}$  (*v*) $\lfloor\underline{4} \times \lfloor\underline{3} = \lfloor\underline{4 \times 3}$  (*vi*) $\dfrac{\lfloor\underline{4}}{\lfloor\underline{3}} = \left\lfloor\underline{\dfrac{4}{3}}\right.$.

**6.** Convert into factorials :

($i$) $3 \times 6 \times 9 \times 12 \times 15 \times 18$ ($ii$) $6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12$.

**7.** Find the value of $n$, if

($i$) $\lfloor n+1 = 20 \lfloor n-1$ ($ii$) $\dfrac{1}{\lfloor 5} + \dfrac{1}{\lfloor 6} = \dfrac{n}{\lfloor 6}$ ($iii$) $\dfrac{1}{\lfloor 6} + \dfrac{1}{\lfloor 7} = \dfrac{n}{\lfloor 8}$ (*NCERT*)

($iv$) $\dfrac{\lfloor n}{\lfloor n-2} = 110$ ($v$) $\dfrac{\lfloor n}{2\lfloor n-2}$ and $\dfrac{\lfloor n}{4\lfloor n-4}$ are in the ratio 1 : 6.

## 7.3 PERMUTATIONS

***Each of the arrangements which can be made by taking some or all of a number of objects is called a permutation.***

For example :

($i$) All permutations *i.e.* arrangements which can be made from the letters $a$, $b$, $c$ by taking 2 at a time are

$ab$, $ba$, $bc$, $cb$, $ca$, $ac$.

Thus, the number of permutations of 3 different objects taken 2 at a time is 6.

($ii$) All permutations which can be made from the letters C, A, T by taking all at a time are

CAT, CTA, ATC, ACT, TAC, TCA.

Thus, the number of permutations of 3 different objects taken all at a time is 6.

*Note that in a permutation the order of arrangement of objects is taken into account. When the order is changed, a different permutation is obtained.*

Thus, in the above example ($i$) $ab$ and $ba$ are different arrangments and in ($ii$) CAT, CTA and ATC etc. are different permutations.

Let $r$ and $n$ be two positive integers and $1 \le r \le n$, then the number of permutations of $n$ different objects taken $r$ at a time, repetition of objects not allowed, is denoted by ${}^nP_r$ or $P(n, r)$.

### 7.3.1 Permutations when all objects are different

**Theorem 1.** ***The number of permutations of n different objects taken r at a time, repetition of objects not allowed, is given by***

$$P(n, r) = n(n-1)(n-2) \ldots (n-r+1) = \frac{\lfloor n}{\lfloor n-r}, \; 1 \le r \le n.$$

**Proof.** The number of permutations of $n$ different objects taken $r$ at a time is the same as the number of ways in which $r$ vacant places

☐ ☐ ☐ ……… ☐

← $r$ vacant places →

can be filled from $n$ different objects.

The first place can be filled up in $n$ different ways because any one of $n$ given objects can be put there. When the first place has been filled up, we are left with $(n-1)$ objects. Any one of them can be put in second place. Similarly, 3rd place can be filled with $n-2$ objects, fourth with $n-3$ objects, …, and $r$th place can be filled in $n-(r-1)$ *i.e.* $n-r+1$ ways. By fundamental principle of counting, $r$ places can be filled up in $n(n-1)(n-2) \ldots (n-r+1)$ ways.

$$\therefore \quad {}^nP_r = n(n-1)(n-2) \ldots (n-r+1)$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)\ldots 3 \times 2 \times 1}{(n-r)\ldots 3 \times 2 \times 1}$$

(Multiplying numerator and denominator by $(n-r)(n-r-1) \ldots 3 \times 2 \times 1$)

$$= \frac{\lfloor n}{\lfloor n-r}, \; 1 \le r \le n.$$

**Corollary 1.** ${}^nP_n = n(n-1) \ldots (n-n+1) = n(n-1) \ldots 1 = \lfloor n$.

**Corollary 2.** Putting $r = n$ in ${}^nP_r = \dfrac{\lfloor n}{\lfloor n-r}$, we get $\lfloor n = \dfrac{\lfloor n}{\lfloor 0}$.

Thus it makes sense to *define* $\lfloor 0 = 1$.

**REMARK**

We define ${}^nP_0 = 1$ *i.e.* the number of permutations of $n$ different objects taken nothing at all is taken as 1.

Counting the number of permutations is merely counting the number of ways in which some or all objects taken at a time are arranged. Arranging no objects at all is the same thing as leaving behind all the objects and we know that there is only one way of doing so. This is why we have defined ${}^nP_0 = 1$.

Thus, the formula ${}^nP_r = \dfrac{\lfloor n}{\lfloor n-r}$ is meaningful for $r = 0$ also.

**Theorem 2.** ***The number of permutations of n different objects taken r at a time, repetition of objects allowed, is $n^r$.***

**Proof.** The number of permutations of $n$ different objects taken $r$ at a time is same as the number of ways in which $r$ vacant places can be filled up from $n$ different objects.

The first place may be filled with any one of $n$ objects. After the first place has been filled up, the second place can also be filled in $n$ ways since we are not prevented from repeating the same object. When the first two places have been filled in $n \times n$ ways, the third place can also be filled up in $n$ ways and so on.

By fundamental principle of counting, $r$ places can be filled up in

$$(n \times n \times \ldots r \text{ times}) \text{ ways } i.e.\ n^r \text{ ways.}$$

Hence, the required number of permutations $= n^r$.

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Find the values of*

*(i)* ${}^5P_5$ *(ii)* ${}^5P_3$ *(iii)* ${}^{60}P_{48}$ *(iv)* ${}^{60}P_2$.

**Solution.** (*i*) ${}^5P_5 = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

(*ii*) ${}^5P_3 = 5 \times 4 \times 3 = 60$.

(*iii*) ${}^{60}P_{48} = \dfrac{\lfloor 60}{\lfloor 60-48} = \dfrac{\lfloor 60}{\lfloor 12}$.

(*iv*) ${}^{60}P_2 = 60 \times 59 = 3540$.

**Example 2.** *Find n if*

*(i)* ${}^nP_4 : {}^nP_5 = 1 : 2$ *(ii)* ${}^nP_4 : {}^{n-1}P_3 = 9 : 1$ *(NCERT)* *(iii)* $P(n, 4) = 20\,P(n, 2)$.

**Solution.** (*i*) ${}^nP_4 : {}^nP_5 = 1 : 2 \Rightarrow \dfrac{n(n-1)(n-2)(n-3)}{n(n-1)(n-2)(n-3)(n-4)} = \dfrac{1}{2}$

$\Rightarrow \quad n - 4 = 2 \Rightarrow n = 6$.

(*ii*) ${}^nP_4 : {}^{n-1}P_3 = 9 : 1 \Rightarrow \dfrac{n(n-1)(n-2)(n-3)}{(n-1)(n-2)(n-3)} = \dfrac{9}{1} \Rightarrow n = 9$.

(*iii*) $P(n, 4) = 20\,P(n, 2) \Rightarrow n(n-1)(n-2)(n-3) = 20\,n(n-1)$

$\Rightarrow \quad (n-2)(n-3) = 20 \Rightarrow n^2 - 5n - 14 = 0 \Rightarrow (n+2)(n-7) = 0$

$\Rightarrow \quad n = -2, 7$. But $n$ cannot be negative, $\therefore \ n = 7$.

**Example 3.** *(i) Find n if P(n, 4) = 2 P(5, 3)* *(ii) Find r if P(10, r) = 720.*

**Solution.** (*i*) Given $P(n, 4) = 2\,P(5, 3)$

$\Rightarrow \quad n(n-1)(n-2)(n-3) = 2.5.4.3$

$\Rightarrow \quad n(n-1)(n-2)(n-3) = 5.4.3.2$

$\Rightarrow \quad n = 5.$

(*ii*) Given $P(10, r) = 720$

$\Rightarrow \quad P(10, r) = 10.9.8$

$\Rightarrow \quad r = 3.$

**Example 4.** *Find n if :*

(*i*) ${}^nP_5 = 42\ {}^nP_3$ *(NCERT)* (*ii*) ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3 : 5.$

**Solution.** (*i*) Given ${}^nP_5 = 42\ {}^nP_3$

$\Rightarrow n(n-1)(n-2)(n-3)(n-4) = 42\, n(n-1)(n-2)$

$\Rightarrow (n-3)(n-4) = 42$

$\Rightarrow n^2 - 7n + 12 = 42$

$\Rightarrow n^2 - 7n - 30 = 0$

$\Rightarrow (n-10)(n+3) = 0$

$\Rightarrow n = 10, -3$ but $n$ cannot be negative,

$\therefore\ n = 10.$

(*ii*) Given ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3 : 5$

$$\Rightarrow \frac{\lfloor\underline{2n+1}}{\lfloor\underline{(2n+1)-(n-1)}} : \frac{\lfloor\underline{2n-1}}{\lfloor\underline{(2n-1)-n}} = 3 : 5$$

$$\Rightarrow \frac{\lfloor\underline{2n+1}}{\lfloor\underline{n+2}} \times \frac{\lfloor\underline{n-1}}{\lfloor\underline{2n-1}} = \frac{3}{5}$$

$$\Rightarrow \frac{(2n+1).2n}{(n+2)(n+1)n} = \frac{3}{5}$$

$\Rightarrow 3(n+2)(n+1) = 10(2n+1)$

$\Rightarrow 3n^2 + 9n + 6 = 20n + 10$

$\Rightarrow 3n^2 - 11n - 4 = 0$

$\Rightarrow (n-4)(3n+1) = 0$

$\Rightarrow n = 4, -\frac{1}{3}$ but $n$ cannot be $-\frac{1}{3}$,

$\therefore\ n = 4.$

**Example 5.** *Find the value of r if*

*(i)* $5\ {}^4P_r = 6\ {}^5P_{r-1}$ *(NCERT)* *(ii)* $P(10, r+1) : P(11, r) = 30 : 11.$

**Solution.** (*i*) Given $5\ {}^4P_r = 6\ {}^5P_{r-1}$

$$\Rightarrow \quad 5 \times \frac{\lfloor\underline{4}}{\lfloor\underline{4-r}} = 6 \times \frac{\lfloor\underline{5}}{\lfloor\underline{5-(r-1)}}$$

$$\Rightarrow \quad \frac{\lfloor\underline{5}}{\lfloor\underline{4-r}} = \frac{6 \times \lfloor\underline{5}}{\lfloor\underline{6-r}} \qquad (\because 5 \times \lfloor\underline{4} = \lfloor\underline{5})$$

$$\Rightarrow \quad \frac{\lfloor\underline{5}}{\lfloor\underline{4-r}} = \frac{6 \times \lfloor\underline{5}}{(6-r)(5-r)\lfloor\underline{4-r}}$$

$\Rightarrow \quad (6-r)(5-r) = 6$

$\Rightarrow \quad 30 - 11r + r^2 = 6$

$\Rightarrow \quad r^2 - 11r + 24 = 0$

$\Rightarrow \quad (r-3)(r-8) = 0$

$\Rightarrow \quad r = 3 \text{ or } r = 8.$

As ${}^4P_r$ and ${}^5P_{r-1}$ are involved, $r \le 4$ and $r - 1 \le 5$

$\Rightarrow \quad r \le 4$ and $r \le 6 \Rightarrow r \le 4$.

Hence rejecting $r = 8$, the required value of $r$ is 3.

(*ii*) Given P (10, $r$ + 1) : P (11, $r$) = 30 : 11

$$\Rightarrow \quad \frac{\lfloor\underline{10}}{\lfloor\underline{10-r-1}} \times \frac{\lfloor\underline{11-r}}{\lfloor\underline{11}} = \frac{30}{11} \Rightarrow \frac{\lfloor\underline{11-r}}{\lfloor\underline{9-r}} = 30$$

$\Rightarrow \quad (11 - r)(10 - r) = 30 \;\Rightarrow r^2 - 21r + 80 = 0$

$\Rightarrow \quad (r - 5)(r - 16) = 0 \;\Rightarrow r = 5, 16.$

But P(10, $r$ + 1), P(11, $r$) are involved, so $r + 1 \le 10$ and $r \le 11$

$\Rightarrow \quad r \le 9$.

Hence rejecting $r = 16$, the required value of $r$ is 5.

**Example 6.** *Prove that (i) P(2n, n) = $2^n$ [1.3.5. … .(2n – 1)]*

*(ii) P(n, r) = n.P(n – 1, r – 1)* *(iii)* ${}^nP_r = {}^{n-1}P_r + r.{}^{n-1}P_{r-1}$.

**Solution.** (*i*) $$P(2n, n) = \frac{\lfloor\underline{2n}}{\lfloor\underline{2n-n}} = \frac{\lfloor\underline{2n}}{\lfloor\underline{n}} = \frac{1.2.3.4.\ \ldots\ .(2n-2)(2n-1)(2n)}{1.2.3.\ \ldots\ .(n-1)n}$$

$$= \frac{[1.3.5.\ \ldots\ .(2n-1)]\,[2.4.6.\ \ldots\ .(2n-2)(2n)]}{1.2.3.\ \ldots\ .(n-1)n}$$

$$= \frac{[1.3.5.\ \ldots\ .(2n-1)].2^n.[1.2.3.\ \ldots\ .(n-1)n]}{1.2.3.\ \ldots\ .(n-1)n} = 2^n\,[1.3.5.\ \ldots\ .(2n-1)].$$

(*ii*) $$\text{R.H.S.} = n.P(n-1, r-1) = n.\frac{\lfloor\underline{n-1}}{\lfloor\underline{n-1-(r-1)}} = \frac{n.\lfloor\underline{n-1}}{\lfloor\underline{n-r}} = \frac{\lfloor\underline{n}}{\lfloor\underline{n-r}}$$

$$= {}^nP_r.$$

(*iii*) $$\text{R.H.S.} = {}^{n-1}P_r + r.{}^{n-1}P_{r-1} = \frac{\lfloor\underline{n-1}}{\lfloor\underline{n-1-r}} + r.\frac{\lfloor\underline{n-1}}{\lfloor\underline{n-1-(r-1)}}$$

$$= \frac{\lfloor\underline{n-1}}{\lfloor\underline{n-r-1}} + r.\frac{\lfloor\underline{n-1}}{\lfloor\underline{n-r}} = \frac{\lfloor\underline{n-1}}{\lfloor\underline{n-r-1}} + \frac{r\lfloor\underline{n-1}}{(n-r)\lfloor\underline{n-r-1}}$$

$$= \frac{\lfloor\underline{n-1}}{\lfloor\underline{n-r-1}}\left(1 + \frac{r}{n-r}\right) = \frac{\lfloor\underline{n-1}}{\lfloor\underline{n-r-1}}.\frac{n-r+r}{(n-r)}$$

$$= \frac{n\lfloor\underline{n-1}}{(n-r)\lfloor\underline{n-r-1}} = \frac{\lfloor\underline{n}}{\lfloor\underline{n-r}} = {}^nP_r.$$

**Example 7.** *How many 4 letter words, with or without meaning, can be formed out of the letters of the word WONDER, if repetition of letters is not allowed ?*

**Solution.** As there are 6 different alphabets in the word WONDER, repetition of letters not allowed, any four of them can be arranged in ${}^6P_4$ ways.

Hence the required number of 4 letter words = ${}^6P_4 = 6 \times 5 \times 4 \times 3 = 360$.

**Example 8.** *How many 4 letter words, with or without meaning, can be formed out of the letters of the word WONDERFUL, if repetition of letters is not allowed?*

**Solution.** As there are 9 different alphabets in the word WONDERFUL, repetition of letters not allowed, any four of them can be arranged in ${}^9P_4$ ways.

Hence the required number of 4 letter words = $9 \times 8 \times 7 \times 6 = 3024$.

# ANSWERS

## EXERCISE 7.1

**1.** 22 **2.** 39 **3.** 374 **4.** 110 **5.** 24

**6.** 12 **7.** 24 **8.** 24 **9.** 120 **10.** 720

**11.** 120 **12.** (*i*) 720 (*ii*) 1000 **13.** 30240

**14.** (*i*) 4 (*ii*) 8 (*iii*) 16 (*iv*) 32

**15.** (*i*) 30 (*ii*) 120 (*iii*) 360 **16.** (*i*) 125 (*ii*) 60

**17.** 4536 **18.** 48 **19.** (*i*) 100 (*ii*) 180

**20.** (*i*) 10 (*ii*) 8 **21.** (*i*) 75 (*ii*) 36

**22.** (*i*) 60 (*ii*) 108 **23.** (*i*) 90 (*ii*) 48

**24.** (*i*) 8 (*ii*) 900 (*iii*) 648 (*iv*) 100 (*v*) 90 (*vi*) 648 (*vii*) 90 (*viii*) 252 **25.** 108 **26.** 81

**27.** 64 **28.** 60 **29.** 620 **30.** 729 **31.** 144; 132

**32.** 17576; 15600 **33.** 225; 210 **34.** 2652; 2704 **35.** (*i*) 40320 (*ii*) 5040

**36.** 720 **37.** 336 **38.** 46800; 26000 **39.** 22814400

**40.** 1800 **41.** 252 **42.** 60 **43.** (*i*) 25 (*ii*) 6

**44.** (*i*) 48 (*ii*) 432 **45.** (*i*) 32 (*ii*) 14

## EXERCISE 7.2

**1.** (*i*) 720 (*ii*) 30 (*iii*) 6 (*iv*) 696 (*v*) 720 (*vi*) 7656 (*vii*) 66 (*viii*) 64 (*ix*) $n + 2$

**2.** (*i*) 24 (*ii*) 120 **3.** (*i*) 30 (*ii*) 5040 (*iii*) 1320 (*iv*) $n(n - 1)$

**4.** (*i*) 10 (*ii*) 455 (*iii*) $\frac{n(n-1)(n-2)}{6}$ **5.** Only (*i*) is true **6.** (*i*) $3^6 \times \lfloor\underline{6}$ (*ii*) $\frac{\lfloor\underline{12}}{\lfloor\underline{5}}$

**7.** (*i*) 4 (*ii*) 7 (*iii*) 64 (*iv*) 11 (*v*) 6

## EXERCISE 7.3

**1.** (*i*) 24 (*ii*) 4160 (*iii*) 2520 (*iv*) 720

**3.** (*i*) 5 (*ii*) 10 (*iii*) 9 (*iv*) 3 (*v*) 8

**4.** (*i*) 30 (*ii*) 120 (*iii*) 360 **5.** (*i*) 504 (*ii*) 729 **6.** 120

**7.** 720 **8.** 5040 **9.** 56 **10.** 11880 **11.** 63 **12.** 256

**13.** $3^{10} - 1$ **14.** 24 **15.** (*i*) $\lfloor\underline{7}$ (*ii*) $\lfloor\underline{5} \times \lfloor\underline{3}$ (*iii*) $\lfloor\underline{5} \times \lfloor\underline{2}$ **16.** 24

**17.** 15 **18.** 1440 **19.** (*i*) 3 (*ii*) 4 (*iii*) 5 (*iv*) 9 (*v*) 5

**20.** (*i*) 24 (*ii*) 120 (*iii*) 720 (*iv*) 5040 (*v*) 40320

**21.** (*i*) 24 (*ii*) 60 (*iii*) 120 (*iv*) 210 (*v*) 336

**22.** 120; 48 **23.** (*i*) 8400 (*ii*) 2520 **24.** (*i*) 32805 (*ii*) 11664 **25.** 27216

**26.** (*i*) 120 (*ii*) 1920 (*iii*) 1956 **27.** (*i*) 40320 (*ii*) 4320 (*iii*) 1440

**28.** (*i*) 207360 (*ii*) 8709120 **29.** (*i*) 48 (*ii*) 72 **30.** 600; 120

**31.** 2880 **32.** 1440 **33.** 86400 **34.** 8640 **35.** 1440

**36.** 4320 **37.** 4200 **38.** (*i*) 720 (*ii*) 120 (*iii*) 24 (*iv*) 240 (*v*) 144

**39.** (*i*) $\lfloor\underline{8} \times \lfloor\underline{7}$ (*ii*) $\lfloor\underline{8} \times \lfloor\underline{7}$ (*iii*) $\lfloor\underline{15}$ **40.** 48 **41.** 12 **42.** 72