8

BINOMIAL THEOREM

INTRODUCTION

In previous classes, you have learnt the squares and cubes of binomial expressions like $a + b$, $a - b$ and used these to find the values of numbers like $(103)^2$, $(998)^3$ by expressing these as $(103)^2 = (100 + 3)^2$, $(998)^3 = (1000 - 2)^3$ etc. However, for higher powers like $(103)^7$, $(998)^9$, the calculations become difficult by repeated multiplication. This problem of evaluation of such numbers was overcome by using a result called **Binomial theorem**. The general form of the binomial expression is $a + b$ and the expansion of $(a + b)^n$, $n \in \mathbf{N}$, is called the **binomial theorem** for positive integral index. **The binomial theorem enables us to expand any power of a binomial expression.** It was first given by Sir Isaac Newton.

Development of Binomial Theorem

We know that

$$(a + b)^0 = 1 \qquad \text{(Assume } a + b \neq 0\text{)}$$
$$(a + b)^1 = a + b$$
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \text{ etc.}$$

From the above expansions, we observe that :

(*i*) The total number of terms in each expansion is one more than the index. For example, in the expansion of $(a + b)^3$, the number of terms is 4 whereas the index of $(a + b)^3$ is 3.

(*ii*) The powers (indices) of the first quantity 'a' goes on decreasing by 1 whereas the powers of the second quantity 'b' goes on increasing by 1, in successive terms.

(*iii*) In each of the expansion, the sum of indices of a and b is the same and is equal to the index of $(a + b)$. For example, in each term of the expansion of $(a + b)^3$, the sum of indices of a and b is 3.

The coefficients of the terms in the above expansions can be written in the form of a table as :

INDEX OF BINOMIAL	COEFFICIENTS OF VARIOUS TERMS
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1

We observe that the coefficients form a certain pattern.

We notice that :

(*i*) each row starts with 1 and ends with 1.

(*ii*) leaving first two rows *i.e.* from third row onwards, each coefficient (except the first and the last) in a row is the sum of two coefficients in the preceding row, one just before it and the other just after it.

The above pattern (arrangement of numbers) is known as **Pascal's Triangle.**

In this pattern, the numbers involved in addition and the results can be indicated as shown in the table below. The table can be extended by writing a few more rows :

INDEX OF BINOMIAL	COEFFICIENTS OF VARIOUS TERMS
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
…	… … … … … … … … … … … … …
…	… … … … … … … … … … … … …

The above table can be continued till any index we like. Expansions for the higher powers of Binomial can be written by using Pascal's triangle. For example, let us expand $(a + b)^6$ by using Pascal's triangle. The row for index 6 is

1 6 15 20 15 6 1

Using this row for coefficients and the observations (*i*), (*ii*) and (*iii*), we get

$$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$$

By making use of the concept of combinations *i.e.* ${}^nC_r = \frac{\lfloor n}{\lfloor n-r\,\lfloor r}$, $0 \le r \le n$, n a non-negative integer, also ${}^nC_n = 1 = {}^nC_0$, the binomial expansions can be written as

$$(a + b)^1 = a + b$$
$$= {}^1C_0\, a^1 + {}^1C_1\, b^1$$
$$(a + b)^2 = a^2 + 2\,ab + b^2$$
$$= {}^2C_0\, a^2 + {}^2C_1\, a^{2-1}\, b^1 + {}^2C_2\, b^2$$
$$(a + b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3$$
$$= {}^3C_0\, a^3 + {}^3C_1\, a^{3-1}\, b^1 + {}^3C_2\, a^{3-2}\, b^2 + {}^3C_3\, b^3 \text{ etc.}$$

By looking at the above expansions, we can easily guess the general formula for the expansion of $(a + b)^n$, $n \in \mathbf{N}$.

8.1 BINOMIAL THEOREM FOR POSITIVE INTEGRAL INDEX

If n is a natural number, a and b are any numbers, then

$$(a + b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \dots + {}^nC_{n-1}ab^{n-1} + {}^nC_nb^n.$$

Proof. We shall prove the theorem by using the principle of mathematical induction.

Let P (n) be the statement :

$$(a + b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}b + {}^nC_2\, a^{n-2}b^2 + \ldots + {}^nC_{n-1}ab^{n-1} + {}^nC_n\, b^n.$$

Here P (1) means

$$(a + b)^1 = {}^1C_0\, a^1 + {}^1C_1 b^1$$

i.e. $\quad a + b = 1 \times a + 1 \times b$, which is true

$\Rightarrow$ P (1) is true.

Let P (m) be true

i.e. $\quad (a + b)^m = {}^mC_0 a^m + {}^mC_1 a^{m-1}b + {}^mC_2 a^{m-2}b^2 + \ldots + {}^mC_{m-1}ab^{m-1} + {}^mC_m b^m \qquad \ldots(i)$

For P (m + 1) :

$$\begin{aligned}(a + b)^{m+1} &= (a + b)^m (a + b)\\ &= ({}^mC_0 a^m + {}^mC_1 a^{m-1}b + {}^mC_2 a^{m-2}b^2 + \ldots + {}^mC_{m-1}ab^{m-1} + {}^mC_m b^m)(a + b) \quad \text{(using } (i))\\ &= {}^mC_0 a^{m+1} + {}^mC_1 a^m b + {}^mC_2 a^{m-1}b^2 + \ldots + {}^mC_{m-1}a^2b^{m-1} + {}^mC_m ab^m\\ &\quad + {}^mC_0 a^m b + {}^mC_1\, a^{m-1}b^2 + {}^mC_2 a^{m-2}b^3 + \ldots + {}^mC_{m-1}ab^m + {}^mC_m\, b^{m+1})\\ &\qquad \text{(by actual multiplication)}\\ &= {}^mC_0 a^{m+1} + ({}^mC_1 + {}^mC_0)\, a^m b + ({}^mC_2 + {}^mC_1)\, a^{m-1}\, b^2 + \ldots\\ &\quad + ({}^mC_m + {}^mC_{m-1})\, ab^m + {}^mC_m b^{m+1} \quad \text{(grouping like terms)}\\ &= {}^{m+1}C_0\, a^{m+1} + {}^{m+1}C_1\, a^m b + {}^{m+1}C_2\, a^{m-1}\, b^2 + \ldots + {}^{m+1}C_m\, ab^m + {}^{m+1}C_{m+1}\, b^{m+1}\end{aligned}$$

(Because we know that ${}^mC_0 = 1 = {}^{m+1}C_0$, ${}^mC_m = 1 = {}^{m+1}C_{m+1}$

and $\quad {}^mC_r + {}^mC_{r-1} = {}^{m+1}C_r$, $r = 1, 2, 3, \ldots, m$

$\Rightarrow \quad {}^mC_1 + {}^mC_0 = {}^{m+1}C_1$, ${}^mC_2 + {}^mC_1 = {}^{m+1}C_2, \ldots, {}^mC_m + {}^mC_{m-1} = {}^{m+1}C_m$)

$\Rightarrow$ P (m + 1) is true.

Hence, by principle of mathematical induction, P (n) is true for all $n \in$ **N**.

The notation $\sum_{r=0}^{n} {}^nC_r\, a^{n-r}\, b^r$ stands for

${}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1}b + {}^nC_2\, a^{n-2}\, b^2 + \ldots + {}^nC_r\, a^{n-r}\, b^r + \ldots + {}^nC_n\, a^{n-n}\, b^n$

(Note that $b^0 = 1$ and $a^{n-n} = a^0 = 1$)

Hence, the **binomial theorem** can be written as

$$\boldsymbol{(a + b)^n = \sum_{r=0}^{n} {}^nC_r\, a^{n-r}\, b^r.}$$

8.1.1 Some important observations

1. The total number of terms in the expansion of $(a + b)^n$ is $(n + 1)$ *i.e.* one more than the index n.
2. The sum of indices of a and b in each term is n. In the first term of the expansion of $(a + b)^n$, the index of a starts with n, goes on decreasing by 1 in every successive term and ends with 0, whereas the index of b starts with zero, goes on increasing by 1 in every successive term and ends with n.
3. The coefficients ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are called **binomial coefficients.**
4. Since ${}^nC_r = {}^nC_{n-r}$, $r = 0, 1, 2, \ldots, n$

   $\Rightarrow {}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$, ${}^nC_2 = {}^nC_{n-2}, \ldots$

   Therefore, the coefficients of terms equidistant from the beginning and end are equal.

8.1.2 Some special cases

1. Replacing 'b' by '$-b$' in the binomial expansion of $(a + b)^n$, we get

$$(a - b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}(-b) + {}^nC_2\, a^{n-2}(-b)^2 + \ldots + {}^nC_r\, a^{n-r}(-b)^r + \ldots + {}^nC_n(-b)^n$$

$$= {}^nC_1\, a^n - {}^nC_1\, a^{n-1} b + {}^nC_2\, a^{n-2} b^2 + \ldots + (-1)^r\, {}^nC_r\, a^{n-r}\, b^r + \ldots + (-1)^n\, {}^nC_n\, b^n$$

$$= \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, a^{n-r}\, b^r$$

Thus, the terms in the expansion of $(a - b)^n$ are alternatively positive and negative. The last term is positive or negative according as n is even or odd.

2. Putting $a = 1$ and $b = x$ in the binomial expansion of $(a + b)^n$, we get

$$(1 + x)^n = {}^nC_0\, 1^n + {}^nC_1\, 1^{n-1} x + {}^nC_2\, 1^{n-2} x^2 + \ldots + {}^nC_r\, 1^{n-r} x^r + \ldots + {}^nC_n\, x^n$$

$$= {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + \ldots + {}^nC_r\, x^r + \ldots + {}^nC_n\, x^n$$

$$= \sum_{r=0}^{n} {}^nC_r\, x^r.$$

3. Putting $a = 1$ and $b = -x$ in the binomial expansion of $(a + b)^n$, we get

$$(1 - x)^n = {}^nC_0 - {}^nC_1\, x + {}^nC_2\, x^2 + \ldots + (-1)^r\, {}^nC_r\, x^r + \ldots + (-1)^n\ {}^nC_n\, x^n$$

$$= \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^r.$$

4. In the expansion of $(1 + x)^n$, $n \in \mathbf{N}$

(*i*) ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_r + \ldots + {}^nC_n = 2^n$.

(*ii*) ${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots + (-1)^n\, {}^nC_n = 0$.

(*iii*) ${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots = 2^{n-1}$.

Proof. We know that

$$(1 + x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + \ldots + {}^nC_r\, x^r + \ldots + {}^nC_n\, x^n \quad \ldots(1)$$

(*i*) On putting $x = 1$ in (1), we get

$$(1 + 1)^n = {}^nC_0 + {}^nC_1 \,.\, 1 + {}^nC_2 \,.\, 1^2 + \ldots + {}^nC_r \,.\, 1^r + \ldots + {}^nC_n \,.\, 1^n$$

$$\Rightarrow\ {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_r + \ldots + {}^nC_n = 2^n.$$

Thus, the sum of the binomial coefficients in the expansion of $(1 + x)^n$, $n \in N$, is 2^n.

(*ii*) On putting $x = -1$ in (1), we get

$$(1 - 1)^n = {}^nC_0 + {}^nC_1\,(-1) + {}^nC_2\,(-1)^2 + {}^nC_3\,(-1)^3 + \ldots + {}^nC_n\,(-1)^n$$

$$\Rightarrow\ {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots + (-1)^n\, {}^nC_n = 0.$$

(*iii*) From part (*ii*), we get

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots .$$

$$\therefore\ \text{The sum of each} = \frac{1}{2}\,(\text{sum of the coefficients of all terms})$$

$$= \frac{1}{2} \,.\, 2^n \qquad \text{(using part } (i))$$

$$= 2^{n-1}$$

$$\Rightarrow\ {}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots = 2^{n-1}.$$

Thus, the sum of the coefficients of odd terms in the expansion of $(1 + x)^n$, $n \in \mathbf{N}$, is equal to the sum of the coefficients of even terms and each is equal to 2^{n-1}.

REMARKS

1. If n is a positive odd integer, then

$(a + b)^n + (a - b)^n$ and $(a + b)^n - (a - b)^n$ both have same number of terms equal to $\frac{n+1}{2}$.

2. If n is a positive even integer, then

(*i*) $(a + b)^n + (a - b)^n$ has $\left(\frac{n}{2}+1\right)$ terms and

(*ii*) $(a + b)^n - (a - b)^n$ has $\frac{n}{2}$ terms.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the number of terms in the expansions of the following :*

(*i*) $(7x + 2y)^9$ (*ii*) $\left(2x - \frac{3}{x^3}\right)^{10}$ (*iii*) $(1 + 2x + x^2)^{11}$

(*iv*) $(x + 2y - 3z)^n$, $n \in \mathbf{N}$.

Solution. (*i*) As the number of terms in the expansion of $(x + a)^n$ is $(n + 1)$, therefore, the number of terms in the expansion of $(7x + 2y)^9 = 9 + 1 = 10$.

(*ii*) The number of terms in the given expansion $= 10 + 1 = 11$.

(*iii*) Given expansion $= (1 + 2x + x^2)^{11} = ((1 + x)^2)^{11} = (1 + x)^{22}$,

$\therefore$ the number of terms in the given expansion $= 22 + 1 = 23$.

(*iv*) Given expansion $= (x + 2y - 3z)^n = (x + (2y - 3z))^n$

$$= {}^nC_0 x^n + {}^nC_1 x^{n-1} (2y - 3z)^1 + {}^nC_2 x^{n-2} (2y - 3z)^2 + \ldots$$
$$+ {}^nC_{n-1} (x)^1 (2y - 3z)^{n-1} + {}^nC_n (2y - 3z)^n.$$

Clearly, the first term in the above expansion gives one term, second term gives 2 terms, third term gives 3 terms and so on, the last term gives $(n + 1)$ terms.

$\therefore$ The total number of terms in the given expansion

$$= 1 + 2 + 3 + \ldots + (n + 1) = \frac{(n+1)(n+2)}{2}.$$

Example 2. *Expand the following :*

(*i*) $(3x - 2y)^4$ (*ii*) $\left(x^2 + \frac{3}{x}\right)^4$, $x \neq 0$. *(NCERT)*

Solution. (*i*) $(3x - 2y)^4 = (3x + (-2y))^4$

$$= {}^4C_0 (3x)^4 + {}^4C_1 (3x)^3 (-2y) + {}^4C_2 (3x)^2 (-2y)^2 + {}^4C_3 (3x)^1 (-2y)^3 + {}^4C_4 (-2y)^4$$
$$= 1.81x^4 + 4.27x^3(-2y) + 6.9x^2.4y^2 + 4.3x(-8y^3) + 1.16y^4$$
$$= 81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4.$$

(*ii*) $$\left(x^2 + \frac{3}{x}\right)^4 = {}^4C_0 (x^2)^4 + {}^4C_1 (x^2)^3 \left(\frac{3}{x}\right) + {}^4C_2 (x^2)^2 \left(\frac{3}{x}\right)^2 + {}^4C_3 x^2 \left(\frac{3}{x}\right)^3 + {}^4C_4 \left(\frac{3}{x}\right)^4$$

$$= x^8 + 4 . x^6 . \frac{3}{x} + 6 . x^4 . \frac{9}{x^2} + 4 . x^2 . \frac{27}{x^3} + \frac{81}{x^4}$$

$$= x^8 + 12x^5 + 54x^2 + \frac{108}{x} + \frac{81}{x^4}.$$

Example 3. *Expand the following :*

(i) $(2x^2 + 3y)^5$ (ii) $\left(\frac{2x^2}{3} - \frac{3}{2x}\right)^4$.

Solution. (i) $(2x^2 + 3y)^5 = {}^5C_0\,(2x^2)^5 + {}^5C_1\,(2x^2)^4(3y) + {}^5C_2\,(2x^2)^3(3y)^2 + {}^5C_3\,(2x^2)^2(3y)^3 + {}^5C_4(2x^2)^1(3y)^4 + {}^5C_5\,(3y)^5$

$= 2^5x^{10} + 5.2^4.3.x^8y + 10.2^3.3^2\,x^6y^2 + 10.2^2.3^3x^4y^3 + 5.2.3^4x^2y^4 + 3^5.y^5$

$= 32x^{10} + 240x^8y + 720x^6y^2 + 1080x^4y^3 + 810x^2y^4 + 243y^5.$

(ii) $$\left(\frac{2x^2}{3} - \frac{3}{2x}\right)^4 = {}^4C_0\left(\frac{2x^2}{3}\right)^4 + {}^4C_1\left(\frac{2x^2}{3}\right)^3\left(\frac{-3}{2x}\right) + {}^4C_2\left(\frac{2x^2}{3}\right)^2\left(\frac{-3}{2x}\right)^2 + {}^4C_3\left(\frac{2x^2}{3}\right)^1\cdot\left(\frac{-3}{2x}\right)^3 + {}^4C_4\left(\frac{-3}{2x}\right)^4$$

$$= \left(\frac{2}{3}\right)^4 x^8 - 4.\left(\frac{2}{3}\right)^3.x^6.\frac{3}{2x} + 6.\left(\frac{2}{3}\right)^2 x^4.\left(\frac{3}{2}\right)^2.\frac{1}{x^2} - 4.\left(\frac{2}{3}\right)^1 x^2.\left(\frac{3}{2}\right)^3.\frac{1}{x^3} + 1.\left(\frac{3}{2}\right)^4.\frac{1}{x^4}$$

$$= \frac{16}{81}x^8 - \frac{16}{9}x^5 + 6x^2 - \frac{9}{x} + \frac{81}{16x^4}.$$

Example 4. *Expand the following :*

(i) $(3x^2 - 2ax + 3a^2)^3$ *(NCERT)* (ii) $(1 - x + x^2)^4$. (*NCERT Examplar Problems*)

Solution. (i) $(3x^2 - 2ax + 3a^2)^3 = (3(x^2 + a^2) - 2ax)^3$

$= {}^3C_0\,(3(x^2 + a^2))^3 - {}^3C_1\,(3(x^2 + a^2))^2\,.\,2ax + {}^3C_2\,3(x^2 + a^2)\,.\,(2ax)^2 - {}^3C_3\,(2ax)^3$

$= 27(x^2 + a^2)^3 - 3\,.\,9(x^2 + a^2)^2\,.\,2ax + 3\,.\,3(x^2 + a^2)\,.\,4a^2x^2 - 8a^3x^3$

$= 27(x^6 + 3x^4a^2 + 3x^2a^4 + a^6) - 54ax(x^4 + 2a^2x^2 + a^4) + 36a^2x^2(x^2 + a^2) - 8a^3x^3$

$= 27x^6 + 81a^2x^4 + 81a^4x^2 + 27a^6 - 54ax^5 - 108a^3x^3 - 54a^5x + 36a^2x^4 + 36a^4x^2 - 8a^3x^3$

$= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6.$

(ii) $(1 - x + x^2)^4 = ((1 - x) + x^2)^4$

$= {}^4C_0(1 - x)^4 + {}^4C_1(1 - x)^3.(x^2)^1 + {}^4C_2(1 - x)^2.(x^2)^2 + {}^4C_3(1 - x)^1.(x^2)^3 + {}^4C_4(x^2)^4$

$= 1.(1 - 4x + 6x^2 - 4x^3 + x^4) + 4(1 - 3x + 3x^2 - x^3)x^2 + 6(1 - 2x + x^2)x^4 + 4(1 - x)\,x^6 + 1.x^8$

$= 1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8.$

Example 5. *Using binomial theorem, find the values of*

(i) $(99)^4$ *(ii)* $(98)^5$ *(NCERT)* *(iii)* $(1.02)^6$ *correct to 5 decimal places.*

Solution. (i) $(99)^4 = (100 - 1)^4 = (10^2 - 1)^4$

$= {}^4C_0(10^2)^4 - {}^4C_1(10^2)^3.1^1 + {}^4C_2(10^2)^2.1^2 - {}^4C_3(10^2)^1.1^3 + {}^4C_1\,1^4$

$= 1.10^8 - 4.10^6 + 6.10^4 - 4.10^2 + 1$

$= 100000000 - 4000000 + 60000 - 400 + 1$

$= 96059601.$

(ii) $(98)^5 = (100 - 2)^5 = (10^2 - 2)^5$

$= {}^5C_0\,(10^2)^5 - {}^5C_1\,(10^2)^4.2 + {}^5C_2\,(10^2)^3.2^2 - {}^5C_3\,(10^2)^2.2^3 + {}^5C_4\,(10^2)^1.2^4 - {}^5C_5\,2^5$

$= 1 \times 10^{10} - 5 \times 10^8 \times 2 + 10 \times 10^6 \times 4 - 10 \times 10^4 \times 8 + 5 \times 10^2 \times 16 - 1 \times 32$

$= 10000000000 - 1000000000 + 40000000 - 800000 + 8000 - 32$

$= 10040008000 - 1000800032$

$= 9039207968.$

(*iii*) $(1.02)^6 = (1 + .02)^6$ $\quad |(1 + x)^n$

$= {}^6C_0 + {}^6C_1\,(.02) + {}^6C_2(.02)^2 + {}^6C_3(.02)^3 + {}^6C_4(.02)^4 + {}^6C_5(.02)^5 + {}^6C_6(.02)^6$

$= 1 + 6(.02) + 15(.0004) + 20(.000008) + 15(.00000016)$

$+ 6(.0000000032) + 1(.000000000064)$

$= 1 + .12 + .006 + .00016 + .0000024 + \ldots$

$= 1.12616$ correct to 5 decimal places.

Example 6. *Which number is larger : $(1.2)^{4000}$ or 800?*

Solution. $(1.2)^{4000} = (1 + 0.2)^{4000}$ $\quad |(1 + x)^n$

$= {}^{4000}C_0 + {}^{4000}C_1(0.2)$ + other positive terms

$= 1 + 4000\,(0.2)$ + other positive terms

$= 1 + 800$ + other positive terms

$> 800.$

Hence, $(1.2)^{4000} > 800$.

Example 7. *Find the value of $(a^2 + \sqrt{a^2 - 1})^4 + (a^2 - \sqrt{a^2 - 1})^4$.* *(NCERT)*

Solution. Let $\sqrt{a^2 - 1} = b$.

$\therefore \quad (a^2 + \sqrt{a^2 - 1})^4 + (a^2 - \sqrt{a^2 - 1})^4 = (a^2 + b)^4 + (a^2 - b)^4$

$= ({}^4C_0\,(a^2)^4 + {}^4C_1\,(a^2)^3b + {}^4C_2\,(a^2)^2b^2 + {}^4C_3\,a^2b^3 + {}^4C_4\,b^4)$

$+ ({}^4C_0\,(a^2)^4 - {}^4C_1\,(a^2)^3b + {}^4C_2\,(a^2)^2b^2 - {}^4C_3\,a^2b^3 + {}^4C_4\,b^4)$

$= 2({}^4C_0\,a^8 + {}^4C_2\,a^4b^2 + {}^4C_4\,b^4)$

$= 2(a^8 + 6a^4(a^2 - 1) + (a^2 - 1)^2)$ (putting the value of b)

$= 2(a^8 + 6a^6 - 6a^4 + a^4 - 2a^2 + 1)$

$= 2(a^8 + 6a^6 - 5a^4 - 2a^2 + 1).$

Example 8. *Expand $(a + b)^6 - (a - b)^6$. Hence find the value of $(\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6$.*

(NCERT)

Solution. $(a + b)^6 - (a - b)^6$

$= ({}^6C_0a^6 + {}^6C_1a^5b + {}^6C_2a^4b^2 + {}^6C_3a^3b^3 + {}^6C_4a^2b^4 + {}^6C_5ab^5 + {}^6C_6b^6)$

$- ({}^6C_0a^6 - {}^6C_1a^5b + {}^6C_2a^4b^2 - {}^6C_3a^3b^3 + {}^6C_4a^2b^4 - {}^6C_5ab^5 + {}^6C_6b^6)$

$= 2\,({}^6C_1a^5b + {}^6C_3a^3b^3 + {}^6C_5ab^5)$

$= 2\,(6\,a^5b + 20\,a^3b^3 + 6\,ab^5)$

$= 4\,ab\,(3\,a^4 + 10\,a^2b^2 + 3\,b^4).$

Putting $a = \sqrt{3}$ and $b = \sqrt{2}$, we get

$(\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6 = 4\sqrt{3}\;\sqrt{2}\;[3(\sqrt{3})^4 + 10(\sqrt{3})^2\,(\sqrt{2})^2 + 3(\sqrt{2})^4]$

$= 4\sqrt{6}\;(3 \times 9 + 10 \times 3 \times 2 + 3 \times 4)$

$= 4\sqrt{6}\;(27 + 60 + 12) = 396\sqrt{6}\,.$

Example 9. *Using binomial theorem, evaluate* $(\sqrt{3}+1)^5 - (\sqrt{3}-1)^5$. *Hence show that the value of* $(\sqrt{3}+1)^5$ *lies between 152 and 153.*

Solution. $(\sqrt{3}+1)^5 - (\sqrt{3}-1)^5$

$$= \left({}^5C_0(\sqrt{3})^5 + {}^5C_1(\sqrt{3})^4 + {}^5C_2(\sqrt{3})^3 + {}^5C_3(\sqrt{3})^2 + {}^5C_4(\sqrt{3})^1 + {}^5C_5\right)$$
$$- \left({}^5C_0(\sqrt{3})^5 - {}^5C_1(\sqrt{3})^4 + {}^5C_2(\sqrt{3})^3 - {}^5C_3(\sqrt{3})^2 + {}^5C_4(\sqrt{3})^1 - {}^5C_5\right)$$
$$= 2({}^5C_1(\sqrt{3})^4 + {}^5C_3(\sqrt{3})^2 + {}^5C_5)$$
$$= 2(5.9 + 10.3 + 1) = 2(45 + 30 + 1) = 152$$

$\Rightarrow \quad (\sqrt{3}+1)^5 = 152 + (\sqrt{3}-1)^5$ ...(*i*)

But we know that

$\sqrt{3} = 1.732 \Rightarrow 0 < \sqrt{3} - 1 < 1$

$\Rightarrow \quad 0 < (\sqrt{3}-1)^5 < 1$ $\quad (\because 0 < a < 1 \Rightarrow 0 < a^n < 1$ for all $n \in \mathbf{N})$

$\therefore$ From (*i*), $(\sqrt{3}+1)^5 = 152 + (\sqrt{3}-1)^5$
= 152 + a positive real number less than 1

$\Rightarrow \quad (\sqrt{3}+1)^5$ lies between 152 and 153.

Example 10. *If P be the sum of odd terms and Q be the sum of even terms in the expansion of* $(x+a)^n$*, prove that*

(*i*) $P^2 - Q^2 = (x^2 - a^2)^n$ (*ii*) $2(P^2 + Q^2) = (x+a)^{2n} + (x-a)^{2n}$

(*iii*) $4PQ = (x+a)^{2n} - (x-a)^{2n}$. (*NCERT Examplar Problems*)

Solution. $(x+a)^n$

$$= {}^nC_0\, x^n + {}^nC_1\, x^{n-1}a + {}^nC_2\, x^{n-2}a^2 + {}^nC_3\, x^{n-3}a^3 + \ldots + {}^nC_n\, a^n$$
$$= ({}^nC_0\, x^n + {}^nC_2\, x^{n-2}a^2 + \ldots) + ({}^nC_1\, x^{n-1}a + {}^nC_3\, x^{n-3}a^3 + \ldots)$$
$$= P + Q \qquad \ldots(1)$$
$$(x-a)^n = {}^nC_0\, x^n - {}^nC_1\, x^{n-1}a + {}^nC_2\, x^{n-2}a^2 - {}^nC_3\, x^{n-3}a^3 + \ldots + {}^nC_n\,(-1)^n a^n$$
$$= ({}^nC_0\, x^n + {}^nC_2\, x^{n-2}a^2 + \ldots) - ({}^nC_1\, x^{n-1}a + {}^nC_3\, x^{n-3}a^3 + \ldots)$$
$$= P - Q \qquad \ldots(2)$$

(*i*) L.H.S. $= P^2 - Q^2 = (P+Q)(P-Q)$
$= (x+a)^n (x-a)^n$ (using (1) and (2))
$= ((x+a)(x-a))^n$
$= (x^2 - a^2)^n$ = R.H.S.

(*ii*) L.H.S. $= 2(P^2 + Q^2) = (P+Q)^2 + (P-Q)^2$
$= ((x+a)^n)^2 + ((x-a)^n)^2$ (using (1) and (2))
$= (x+a)^{2n} + (x-a)^{2n}$.

(*iii*) L.H.S. $= 4PQ = (P+Q)^2 - (P-Q)^2$
$= ((x+a)^n)^2 - ((x-a)^n)^2$ (using (1) and (2))
$= (x+a)^{2n} - (x-a)^{2n}$.

Example 11. *Write the binomial expansion of* $(1+x)^{n+1}$*, when* $x = 8$*. Deduce that* $9^{n+1} - 8n - 9$ *is divisible by 64 for all* $n \in \mathbf{N}$. *(NCERT)*

Solution. By binomial theorem,

$$(1+x)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1 x + {}^{n+1}C_2 x^2 + {}^{n+1}C_3 x^3 + \ldots + {}^{n+1}C_{n+1} x^{n+1}.$$

Putting $x = 8$, we get

$(1+8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1 8 + {}^{n+1}C_2 8^2 + {}^{n+1}C_3 8^3 + \ldots + {}^{n+1}C_{n+1} 8^{n+1}$, which is the required binomial expansion of $(1+x)^{n+1}$ when $x = 8$

$\Rightarrow \quad 9^{n+1} = 1 + (n + 1)\, 8 + {}^{n+1}C_2 8^2 + {}^{n+1}C_3 8^3 + \ldots + {}^{n+1}C_{n+1} 8^{n+1}$

$(\because\ {}^{n+1}C_0 = 1 \text{ and } {}^{n+1}C_1 = n + 1)$

$\Rightarrow \quad 9^{n+1} - 8n - 9 = 64({}^{n+1}C_2 + {}^{n+1}C_3 8 + \ldots + {}^{n+1}C_{n+1} 8^{n-1})$

$= 64\lambda$, where λ is some integer

$\Rightarrow$ $9^{n+1} - 8n - 9$ is divisible by 64 for all $n \in \mathbf{N}$.

Example 12. *By using Binomial theorem, prove that $3^{2n+2} - 8n - 9$ is divisible by 64 for all natural numbers n.*

Solution. We have, $3^{2n+2} - 8n - 9 = (3^2)^{n+1} - 8n - 9$

$= (1 + 8)^{n+1} - 8n - 9$

$= (1 + {}^{n+1}C_1\, 8 + {}^{n+1}C_2\, 8^2 + {}^{n+1}C_3\, 8^3 + \ldots + {}^{n+1}C_{n+1}\, 8^{n+1}) - 8n - 9$

$= 1 + (n + 1) \times 8 + 8^2 ({}^{n+1}C_2 + {}^{n+1}C_3\, 8 + \ldots + {}^{n+1}C_{n+1}\, 8^{n-1}) - 8n - 9$

$= 9 + 8n + 64 ({}^{n+1}C_2 + {}^{n+1}C_3\, 8 + \ldots + {}^{n+1}C_{n+1}\, 8^{n-1}) - 8n - 9$

$= 64\lambda$, where λ is some integer

$\Rightarrow$ $3^{2n+2} - 8n - 9$ is divisible by 64 for all natural numbers n.

Example 13. *Using binomial theorem, prove that $6^n - 5n$ always leaves the remainder 1 when divided by 25, for all $n \in$ **N**.* *(NCERT)*

Solution. We have, $6^n - 5n = (1 + 5)^n - 5n$ $\quad | (1 + x)^n$

$= ({}^nC_0 + {}^nC_1 \times 5 + {}^nC_2 \times 5^2 + {}^nC_3 \times 5^3 + \ldots + {}^nC_n \times 5^n) - 5n$

$= 1 + 5n + {}^nC_2 \times 5^2 + {}^nC_3 \times 5^3 + \ldots + {}^nC_n \times 5^n - 5n$

$= 1 + 5^2({}^nC_2 + {}^nC_3 \times 5^1 + \ldots + {}^nC_n \times 5^{n-2})$

$= 1 + 25\lambda$, where $\lambda = {}^nC_2 + {}^nC_3 \times 5 + \ldots + {}^nC_n \times 5^{n-2}$ is some integer.

Thus, $6^n - 5n = 25\lambda + 1$, where λ is some integer

$\Rightarrow$ $6^n - 5n$ leaves the remainder 1 when divided by 25.

EXERCISE 8.1

Very short answer type questions (1 to 4) :

1. Find the number of terms in the expansions of the following :

(*i*) $\left(3x - \dfrac{7}{y^2}\right)^8$ (*ii*) $(1 + 2x + x^2)^7$ (*iii*) $(x^2 - 6x + 9)^{10}$.

2. Find the number of terms in the expansion of the following :

(*i*) $(1 + 3x + 3x^2 + x^3)^5$ (*ii*) $(a - b + c)^6$.

3. Find the number of terms in the expansions of the following :

(*i*) $(2x + 3y)^{49} + (2x - 3y)^{49}$ (*ii*) $(\sqrt{3} + 5x^2)^{93} - (\sqrt{3} - 5x^2)^{93}$.

4. Find the number of terms in the expansions of the following :

(*i*) $(4x^2 + 5\sqrt{3}\,y)^{100} + (4x^2 - 5\sqrt{3}\,y)^{100}$

(*ii*) $(1 + 7\sqrt{2}\,x)^{50} - (1 - 7\sqrt{2}\,x)^{50}$.

5. By using binomial theorem, expand the following :

(*i*) $(2x + 3y)^5$ (*ii*) $(1 - 2x)^5$ (*NCERT*) (*iii*) $\left(\dfrac{2x}{3} - \dfrac{3}{2x}\right)^4$

(*iv*) $\left(\dfrac{2}{x} - \dfrac{x}{2}\right)^5$ (*NCERT*) (*v*) $\left(\dfrac{x}{3} + \dfrac{1}{x}\right)^5$ (*NCERT*) (*vi*) $(2x - 3)^6$ (*NCERT*) (*vii*) $\left(x + \frac{1}{x}\right)^6$ (*NCERT*) (*viii*) $(1 + x + x^2)^3$.

6. Using binomial theorem, find the values of :

(*i*) $(96)^3$ (*NCERT*) (*ii*) $(101)^4$ (*NCERT*) (*iii*) $(102)^5$ (*NCERT*)

(*iv*) $(99)^5$ (*NCERT*) (*v*) $(999)^3$ (*vi*) $(10.1)^4$.

7. (*i*) Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

(*ii*) Find the value of $(1.01)^5$ correct to 5 decimal places.

8. Which number is larger

(*i*) $(1.1)^{10000}$ or 1000 ? (*NCERT*) (*ii*) $(1.01)^{1000000}$ or 10000 ? (*NCERT*)

9. Simplify the following :

(*i*) $(x^2 - \sqrt{1-x^2})^4 + (x^2 + \sqrt{1-x^2})^4$ (*NCERT Examplar Problems*)

(*ii*) $(x + \sqrt{x-1})^6 + (x - \sqrt{x-1})^6$

10. Using binomial theorem, evaluate the following :

(*i*) $(\sqrt{3} + \sqrt{2})^3 + (\sqrt{3} - \sqrt{2})^3$ (*ii*) $(\sqrt{3} + 1)^5 - (\sqrt{3} - 1)^5$

(*iii*) $(3 + \sqrt{2})^5 - (3 - \sqrt{2})^5$ (*iv*) $(2 + \sqrt{3})^7 + (2 - \sqrt{3})^7$.

11. Find $(a + b)^4 - (a - b)^4$. Hence evaluate $(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4$. (*NCERT*)

12. Using Binomial theorem, expand $(x + y)^5 + (x - y)^5$. Hence find the value of

$(\sqrt{2} + 1)^5 + (\sqrt{2} - 1)^5$.

13. Using Binomial theorem, find $(x + 1)^6 + (x - 1)^6$. Hence or otherwise evaluate

$(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6$. (*NCERT*)

14. Using Binomial theorem, expand $(a + b)^4$. Hence or otherwise prove that

$(1 - x)^8 = (1 - 2x)^4 + 4x^2(1 - 2x)^3 + 6x^4(1 - 2x)^2 + 4x^6(1 - 2x) + x^8$.

Hint. $(1 - x)^8 = ((1 - x)^2)^4 = ((1 - 2x) + x^2)^4$.

15. In the Binomial expansion of $(\sqrt[3]{3} + \sqrt{2})^5$, find the term which does not contain irrational expression.

16. Prove that $\sum_{r=0}^{n} 3^r\, {}^nC_r = 4^n$. (*NCERT*)

Hint. $(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_r x^r + \ldots + {}^nC_n x^n$. Put $x = 3$.

17. By using binomial theorem, prove that :

(*i*) $2^{3n} - 7n - 1$ is divisible by 49, for all $n \in \mathbf{N}$.

(*ii*) $3^{3n} - 26n - 1$ is divisible by 676, for all $n \in \mathbf{N}$.

Hint. (*i*) $2^{3n} - 7n - 1 = (2^3)^n - 7n - 1 = 8^n - 7n - 1 = (1 + 7)^n - 7n - 1$.

(*ii*) $3^{3n} - 26n - 1 = (3^3)^n - 26n - 1 = (1 + 26)^n - 26n - 1$.

8.2 GENERAL AND MIDDLE TERMS

8.2.1 General term

If n is any natural number and a, b are any numbers, then

$$(a + b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1} b + {}^nC_2\, a^{n-2} b^2 + \ldots + {}^nC_r\, a^{n-r} b^r + \ldots + {}^nC_n\, b^n.$$

In the binomial expansion of $(a + b)^n$, we find that the first term is ${}^nC_0\, a^n$, second term is ${}^nC_1\, a^{n-1}\, b$, the third term is ${}^nC_2\, a^{n-2}\, b^2$ and so on. On looking at the pattern of sucessive terms, we find that $(r + 1)$th term is ${}^nC_r\, a^{n-r}\, b^r$. It is denoted by T_{r+1}.

Thus, $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$. This is called the **general term.**

Hence, general term = $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$.

Particular cases

(*i*) In the expansion of $(a - b)^n$, $T_{r+1} = (-1)^r\, {}^nC_r\, a^{n-r}\, b^r$.

(*ii*) In the expansion of $(1 + x)^n$, $T_{r+1} = {}^nC_r\, x^r$.

(*iii*) In the expansion of $(1 - x)^n$, $T_{r+1} = (-1)^r\, {}^nC_r\, x^r$.

REMARKS

1. Coefficient of x^r in the expansion of $(1 + x)^n$ is ${}^nC_r$.
2. rth term from the end in the expansion of $(a + b)^n$ is rth term from the beginning in the expansion of $(b + a)^n$.

   Alternatively, as the total number of terms in the expansion of $(a + b)^n$ is $n + 1$,

   $\therefore$ rth term from end has $((n + 1) - r)$ *i.e.* $(n - r + 1)$ terms before it, therefore, it is $(n - r + 2)$th term from beginning.

8.2.2 Middle term or terms

Since the binomial expansion of $(a + b)^n$ contains $(n + 1)$ terms, therefore

(*i*) if n is even then the number of terms in the expansion is odd, so there is only one middle term and $\left(\frac{n}{2} + 1\right)$th term *i.e.* $T_{\frac{n}{2}+1}$ is the middle term.

(*ii*) if n is odd then the number of terms in the expansion is even, so there are two middle terms and $\left(\frac{n+1}{2}\right)$th, $\left(\frac{n+1}{2} + 1\right)$ th *i.e.* $T_{\frac{n+1}{2}}$, $T_{\frac{n+3}{2}}$ are the two middle terms.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the 7th term in the expansion of* $\left(2x^3 - \frac{3}{2x}\right)^{10}$.

Solution. We know that in the expansion of $(a + b)^n$, $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$.

$\therefore$ In the expansion of $\left(2x^3 - \frac{3}{2x}\right)^{10}$,

$$T_7 = T_{6+1} = {}^{10}C_6 (2x^3)^{10-6} \left(-\frac{3}{2x}\right)^6 = {}^{10}C_4 (2x^3)^4 \left(\frac{3}{2x}\right)^6$$

$$= \frac{10.9.8.7}{1.2.3.4} \cdot 2^4 \cdot x^{12} \cdot \frac{3^6}{2^6 x^6} = \frac{76545}{2} x^6.$$

Example 2. *Find the 13th term in the expansion of* $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$, $x > 0$. *(NCERT)*

Solution. In the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$,

$$T_{13} = T_{12+1} = {}^{18}C_{12} (9x)^{18-12} \left(-\frac{1}{3\sqrt{x}}\right)^{12}$$

$$= {}^{18}C_6 (9x)^6 \times \frac{1}{3^{12} x^6} \qquad (\because {}^nC_r = {}^nC_{n-r})$$

$$= \frac{18 \times 17 \times 16 \times 15 \times 14 \times 13}{1 \times 2 \times 3 \times 4 \times 5 \times 6} \times (3^2)^6 x^6 \times \frac{1}{3^{12} x^6}$$
$$= 18564.$$

Example 3. *Find the fourth term from the end in the expansion of* $\left(\frac{3}{x^2} - \frac{x^3}{3}\right)^9$.

Solution. The fourth term from the end in the expansion of $\left(\frac{3}{x^2} - \frac{x^3}{3}\right)^9$

= the fourth term from the beginning in the expansion of $\left(-\frac{x^3}{3} + \frac{3}{x^2}\right)^9$.

(Interchanging a and b in $(a + b)^n$)

$$\therefore \quad T_4 = T_{3+1} = {}^9C_3 \left(-\frac{x^3}{3}\right)^{9-3} \left(\frac{3}{x^2}\right)^3 = \frac{9.8.7}{1.2.3} \cdot \left(-\frac{x^3}{3}\right)^6 \cdot \left(\frac{3}{x^2}\right)^3$$
$$= 84 \cdot \frac{x^{18}}{3^6} \cdot \frac{3^3}{x^6} = 84 \cdot \frac{x^{12}}{3^3} = \frac{28}{9} x^{12}.$$

Alternatively

4th term from the end = (9 – 4 + 2)th *i.e.* 7th term from the beginning of the given expansion.

$$\therefore \text{ 4th term from the end } = {}^9C_6 \left(\frac{3}{x^2}\right)^{9-6} \left(-\frac{x^3}{3}\right)^6 = {}^9C_3 \left(\frac{3}{x^2}\right)^3 \left(\frac{x^3}{3}\right)^6$$
$$= \frac{9.8.7}{1.2.3} \cdot \frac{3^3}{x^6} \cdot \frac{x^{18}}{3^6} = \frac{28}{9} x^{12}.$$

Example 4. *Find the rth term from the end in the expansion of* $(x + a)^n$, $n \in \mathbf{N}$. *(NCERT)*

Solution. The rth term from the end in the expansion of $(x + a)^n$

= the rth term from the beginning in the expansion of $(a + x)^n$

$= T_r = T_{(r-1)+1} = {}^nC_{r-1}\, a^{n-(r-1)}\, x^{r-1}$

$= {}^nC_{r-1}\, x^{r-1}\, a^{n-r+1}$.

Example 5. *Find x if the 17th and 18th terms of the expansion* $(2 + x)^{50}$ *are equal.* *(NCERT)*

Solution. In the expansion of $(2 + x)^{50}$,

$T_{17} = T_{16+1} = {}^{50}C_{16}\, 2^{50-16}\, x^{16}$ and

$T_{18} = T_{17+1} = {}^{50}C_{17}\, 2^{50-17}\, x^{17}$.

Given $T_{17} = T_{18} \Rightarrow {}^{50}C_{16}\, 2^{34}\, x^{16} = {}^{50}C_{17}\, 2^{33}\, x^{17}$

$$\Rightarrow \quad \frac{\lfloor 50}{\lfloor 34 \lfloor 16} \times 2 = \frac{\lfloor 50}{\lfloor 33 \lfloor 17} \times x$$

$$\Rightarrow \quad \frac{2}{34 \times \lfloor 33 \times \lfloor 16} = \frac{x}{\lfloor 33 \times 17 \times \lfloor 16} \Rightarrow \frac{2}{34} = \frac{x}{17}$$

$$\Rightarrow \quad x = 1.$$

Example 6. *Find the middle term in* $\left(\frac{2x^2}{3} - \frac{3}{2x}\right)^{12}$.

Solution. Total number of terms in the given expansion = 12 + 1 = 13 (odd).

$\therefore$ There is only one middle term given by $T_{\frac{12}{2}+1}$ *i.e.* T_7.

ANSWERS

EXERCISE 8.1

1. (*i*) 9 (*ii*) 15 (*iii*) 21 **2.** (*i*) 16 (*ii*) 28

3. (*i*) 25 (*ii*) 47 **4.** (*i*) 51 (*ii*) 25

5. (*i*) $32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$

(*ii*) $1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$ (*iii*) $\frac{16}{81}x^4 - \frac{16}{9}x^2 + 6 - \frac{9}{x^2} + \frac{81}{16x^4}$

(*iv*) $\frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5x^3}{8} - \frac{x^5}{32}$ (*v*) $\frac{x^5}{243} + \frac{5x^3}{81} + \frac{10x}{27} + \frac{10}{9x} + \frac{5}{3x^3} + \frac{1}{x^5}$

(*vi*) $64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$

(*vii*) $x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$

(*viii*) $x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1$

6. (*i*) 884736 (*ii*) 104060401 (*iii*) 11040808032

(*iv*) 9509900499 (*v*) 997002999 (*vi*) 10406.0401

7. (*i*) 0.951 (*ii*) 1.05101 **8.** (*i*) $(1.1)^{10000}$ (*ii*) $(1.01)^{1000000}$

9. (*i*) $2x^8 - 12x^6 + 14x^4 - 4x^2 + 2$ (*ii*) $2(x^6 + 15x^5 - 29x^3 + 12x^2 + 3x - 1)$

10. (*i*) $18\sqrt{3}$ (*ii*) 152 (*iii*) $1178\sqrt{2}$ (*iv*) 10084 **11.** $8ab(a^2 + b^2)$; $40\sqrt{6}$

12. $2(x^5 + 10x^3y^2 + 5xy^4)$; $58\sqrt{2}$ **13.** $2(x^6 + 15x^4 + 15x^2 + 1)$; 198

14. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ **15.** Third term = 60

EXERCISE 8.2

1. (*i*) $(-1)^r\ {}^6C_r\ x^{12-2r}y^r$ (*ii*) $(-1)^r\ {}^{12}C_r\ x^{24-r}\ y^r$ (*iii*) $(-1)^r\ {}^{12}C_r\ x^{24-3r}$

2. 112.3^6x^{10} **3.** $-1760\ x^9y^3$ **4.** $\frac{1760}{x^3}$ **5.** $\frac{672}{x^3}$ **6.** ${}^{10}C_5$

7. (*i*) $-{}^{10}C_5$ (*ii*) $-20x^3$ **8.** ${}^{2n}C_n\,x^n$ **9.** 2, –2 **10.** –3

11. –4 **12.** 4 **13.** 8th **14.** 7th **15.** 2^{50}

16. 2^{49} **17.** 63 **18.** (*i*) $-\frac{\lfloor 25}{\lfloor 15\,\lfloor 10}\cdot\frac{2^{10}}{x^{20}}$ (*ii*) $\frac{\lfloor 3n}{\lfloor n\,\lfloor 2n}\cdot\frac{1}{x^n}$

19. $\frac{7}{8}$

20. (*i*) $61236x^5y^5$ (*ii*) $-\frac{105}{8}x^9, \frac{35}{48}x^{12}$ (*iii*) $\frac{59136a^6b^6}{x^6}$ (*iv*) $\frac{189}{8}x^{17}, -\frac{21}{16}x^{19}$

22. (*i*) 1512 (*ii*) –252 (*iii*) $55.2^8.3^5$ (*iv*) 0 (*v*) –25344

23. $-9720; -\frac{40}{27}$

24. (*i*) 672 (*ii*) 924 (*iii*) 0 **25.** –438 **26.** 990

27. (*i*) –3432 (*ii*) 495 (*iii*) $\frac{5}{12}$ (*iv*) $-3003 \times 3^{10} \times 2^5$ **28.** 4 **30.** $\frac{9}{7}$

32. 6 **33.** 1, 14 **34.** 15 **35.** $3003\ y^4x^{10}$

36. 30th and 31st terms **37.** 8 **38.** 11; $462x^7$

39. $a = 2, n = 4$ **40.** $n = 11, x = 2$ **41.** $(1 + 2)^5$ **42.** $(3 + 5)^6$ **43.** 12

44. $n = 7, r = 3$ **45.** 7